COMPLETE FINITE FROBENIUS GROUPS AND WREATH PRODUCTS

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Abstract

A group G is complete if it has a trivial center and for every automorphism φ of G there is an element $x \in G$ such that $\varphi(g) = x^{-1}gx$ for all $g \in G$. H. Wielandt and J. S. Rose proved that every finite group G can be embedded as a subnormal subgroup in a finite complete group K, meaning that there exists a sequence of subgroups $\{G_i\}$ of K such that

$$G = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_i \trianglelefteq \cdots \trianglelefteq G_n = K.$$

Moreover if G is solvable then K can be solvable too. This dissertation classifies complete Frobenius groups and complete finite permutational wreath products, in addition to investigating the structure of an odd-order complete group.

We also show that in a finite permutational wreath product $G \wr H$, if the base group is not characteristic then G is the semidirect product of an odd-order abelian group of index 2 with a cyclic group of order 2 acting by inversion. In the case where H acts transitively we provide a biconditional statement and determine that H is also a wreath product, which confirms earlier results by P. Neumann and Y. V. Bodnarchuk. Lastly we investigate the structure of the automorphism group of $G \wr H$ when the base is not characteristic.

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Contents

Introduction

A group G is said to be *complete* if its center is trivial and for every automorphism φ of G there exists an element $x \in G$ so that $\varphi(g) = x^{-1}gx$ for all $g \in G$. The assignment for this dissertation was to answer the following question: "For a given finite group G, what's the smallest finite complete group K in which G can be embedded?" Since there is a well-understood algorithm, discussed in Chapter 3.1, for embedding a finite group in a finite complete group, part of the question was to determine a useful way of defining "smallest."

In short order the focus turned to constructing a complete group with a given normal subgroup and then to the task of linking the automorphisms of a group to those of a normal subgroup. The reader will notice that while the following chapters on the surface may seem somewhat unrelated, what truly connects the topics is the drive to understand how automorphisms can be constructed and how the behavior of a known automorphism dictates the structure of the group.

In the first chapter we set the foundation for studying automorphisms in the contexts of this research. In Chapter 2 we define a group construction known as the permutational wreath product of G with H and explore how the automorphisms of the wreath product determine and are determined by the structure of the components. We return to complete groups in Chapter 3, starting with a brief history of the study of complete groups and the motivation behind the study. We develop an approach, using cohomology of groups, that allows us to classify complete wreath products and Frobenius groups. Additionally we explore the structure of an odd-order complete group.

Throughout this work we will maintain the following notation. Unless otherwise stated, all groups discussed here will be finite. Our notation with regards to standard examples and terminology will be consistent with that used in [9] unless otherwise described. For example, we use $\mathrm{Z}(G)$ to represent the center of a group G. For the remainder of our discussion of notation, let G be a group with $N \trianglelefteq G$ and $H \leq G$.

Permutations will act on the right, whereas automorphisms will be written on the left or as superscripts on the right. Notice that if $g \in G$ and φ, ψ are two automorphisms of G then

$$\begin{aligned} \psi\varphi(g) &= \psi(g^{\varphi}) \\ &= (g^{\varphi})^{\psi} \\ &= g^{\varphi\psi}. \end{aligned}$$

If automorphisms are written on the left then composition is performed from right to left. If automorphisms are written as superscripts then composition will be performed from left to right.

If G has an automorphism φ that fixes N set-wise, then we will write the restriction of φ to N as $\varphi_{|N}$. On the other hand if φ is an automorphism of $H \leq G$ that can be extended to an automorphism of G then we will write φ^* for the extension. The collection of all automorphisms of G will be denoted by Aut G. Our notation regarding automorphisms is discussed further in Chapter 1.

The symmetric group of degree n will be denoted by S_n and the group of all $n \times n$ matrices with non-zero determinant and entries in the field of order p will be written as $\mathrm{GL}_n(p)$. We assume that the reader is familiar with the dihedral group of order 8, the quaternion group, and the cyclic group of order n. We shall denote these groups as D_8, Q_8, and C_n, respectively.

We further assume that the reader is familiar with the concepts and results from introductory group theory, such as direct products and the statement of Sylow's Theorems ([9, Theorem 4.5.18]). When a more advanced topic is discussed we will provide the relevant background information and direct the reader to a appropriate reference text for details.

The subgroup H is a *complement of* N if $G = NH$ and $N \cap H = 1$. In this case G is the *semidirect product of* N *with* H and we write $G = N \rtimes H$. In general, the semidirect product of N with H can be formed if there exists a homomorphism $\psi : H \to \mathrm{Aut}\ N$. Every element $g \in N \rtimes H$ can be written uniquely as nh for some $n \in N$ and $h \in H$. For $n_1, n_2 \in N$ and $h_1, h_2 \in H$ multiplication in the semidirect product $N \rtimes H$ is calculated as follows:

$$n_1h_1 \cdot n_2h_2 = n_1n_2^{\psi(h_1^{-1})}h_1h_2.$$

If G is a finite semidirect product of N with H then $|G| = |N||H|$.

One special type of semidirect product is a holomorph. For any group N let the H described in the previous section be Aut N. Then $G = N \rtimes H$ is known as the *holomorph of* N. In this group an element $\varphi \in H$ conjugates an element $n \in N$ in the same way that it acts on n, so that

$$n^{\varphi} = \varphi(n).$$

This gives rise to our decision to write automorphisms as superscripts on the right.

The subgroup H is called *subnormal in* G if there exists an integer n and a sequence of subgroups $H_i \trianglelefteq G$ such that

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_i \trianglelefteq \cdots \trianglelefteq H_n = G.$$

When H is subnormal we say H has *defect* n if n is the smallest integer for which such a sequence exists. Notice that all proper normal subgroups of G are subnormal subgroups of defect 1.

For two groups, G and K, we say that G is *embedded as a normal subgroup* in K if there exists an injective homomorphism $\varphi : G \to K$ with the image of G a normal subgroup of K. Similarly G is *embedded as a subnormal subgroup* in K if G is isomorphic to a subnormal subgroup of K.

We refer to a group whose order is a power of a prime p as a *p-group*. A group is an *elementary abelian p-group* if it is the direct product of some number of copies of C_p, for a fixed prime p. The smallest subgroup of G with a 2-group quotient will be denoted by $\mathbb{O}^2(G)$. On the other hand, the largest normal 2-subgroup will be written as $\mathbb{O}_2(G)$. Every p-group is also a *nilpotent group*, which are groups that are isomorphic to a direct product of their Sylow p-subgroups. We shall use the facts that the center of a non-trivial nilpotent group is always nontrivial and that every proper subgroup of a nilpotent group is a proper subgroup of its normalizer. An introduction to p-groups and nilpotent groups can be found in Chapter 6 of [9]. We will, however, need to refer to the largest nilpotent normal subgroup of G, called the *Fitting subgroup of* G, which will be denoted by Fit (G).

To generalize the idea of a Sylow p-subgroup of a group G, we say that H is a *Hall subgroup* of G if $|H|$ and $|G : H|$ are relatively prime. When π is a set containing all prime divisors of $|H|$ and no prime divisors of $|G : H|$ then we call H a *Hall π-subgroup* of G. As with Sylow subgroups, a normal Hall π-subgroup is characteristic. However, the remaining parts of Sylow's Theorem [9, Theorem 4.5.18] do not generalize – there is no guarantee in a general group G that a Hall π-subgroup exists if π has more than one prime in it, nor will the existing Hall π-subgroups for a fixed π necessarily be conjugates of one another.

Let $g_1, g_2 \in G$. The *commutator of* g_1 *with* g_2 is the element $g_1^{-1}g_2^{-1}g_1g_2$, which will be represented by $[g_1, g_2]$. The *commutator subgroup* of G is the group generated by possible commutators and will be written G'. Another term for this characteristic subgroup is "*derived subgroup*" of G. The *derived series* of G is the series defined recursively with $G_0 = G$ and $G_{i+1} = G'_i$. Each element in the series is a normal subgroup of G. If there exists a natural number n such that $G_n = 1$ then we say that G is *solvable*. Equivalently G is solvable if there exists a series $\{G_i\}$ of subgroups of G with

$$1 = G_0 \trianglelefteq \cdots \trianglelefteq G_i \trianglelefteq \cdots \trianglelefteq G_n = G$$

where every quotient G_i/G_{i-1} is abelian.

We will discuss solvable groups in greater depth in Chapter 3.2. Readers who are interested in the study of solvable groups should refer to [22] or [37] for an introduction. The text by Doerk and Hawkes [8] provides a complete resource for an advanced study of solvable groups.

Before moving on to discuss our results in detail, we would like to thank Rex Dark for suggesting a reading list of papers about examples of complete odd-order groups. Thanks also to Marion Deaconescu, who sent us in the direction of [25] and an understanding of complete holomorphs. We thank our collaborator, Donald Passman, for his work on the results in Chapter 2. Additionally, we'd like to thank Derek J. S. Robinson, who was gracious in his support of this research and initially suggested the proofs for Theorem 3.3.19 and Proposition 3.3.20.

Chapter 1

Automorphisms

This chapter presents the ideas from the study of group automorphisms upon which the later chapters build. We begin with notation and the definition of some standard concepts in the first section. The second section discusses constructing automorphisms from those of the components in a semidirect product. The chapter closes with an introduction to how cohomology of groups can be used in the study of automorphisms.

1.1 Definitions and Notation

We denote the group of automorphisms of a group G by Aut G. Recall that a fixed element $x \in G$ defines an automorphism τ_x of G through conjugation:

$$\tau_x(g) = x^{-1}gx \qquad \text{for all } g \in G.$$

Automorphisms defined in this manner are called *inner automorphisms of G* and the collection of inner automorphisms is denoted as Inn G. Notice that by our notation $\tau_y \tau_x(g) = \tau_{xy}(g) = g^{xy} = g^{\tau_{xy}}$.

Proposition 1.1.1. *Let G be a group. Then Inn G is a normal subgroup of Aut G.*

The proof of Proposition 1.1.1 is straightforward; often it is a homework problem in graduate algebra classes. The statement can be found in any introductory abstract algebra text. For example, see Chapter 4.4 of [9].

The quotient group Aut $G/$Inn G is referred to as the *outer automorphism group of G* and its elements are called *outer automorphisms.* Some authors refer to automorphisms that are not elements of Inn G as outer automorphisms but we reserve the term purposefully for the elements of the quotient. Instead we refer to automorphisms that are not members of Inn G as *non-inner* automorphisms.

The term "inner automorphism" no doubt arises from the following proposition.

Proposition 1.1.2. *Let G be a group. The map from G into Inn G defined by $g \mapsto \tau_g$ is a group homomorphism with kernel $Z(G)$.*

Proposition 1.1.2 is also a well-known fact from graduate algebra courses. The statement and its proof can be found in just about any introductory abstract algebra textbook. Note that if the center of G is trivial then the described homomorphism is in fact an isomorphism, giving the next corollary.

Corollary 1.1.3. *Let G be a group with trivial center. Then G can be embedded as a normal subgroup in Aut G with image Inn G.*

For some groups, the automorphism group has been completely determined. This is the case with cyclic groups, elementary abelian groups, and the alternating groups, for example. The next proposition summarizes some results of this type.

Theorem 1.1.4. *Let p be a prime and n a positive integer.*

1. *[9, Proposition 4.4.16] The automorphism group of C_n is isomorphic to an abelian group of order m where $m = \varphi(n)$, the value of Euler's function on n.*
2. *[9, Proposition 4.4.17(1)] The automorphism group of C_p is isomorphic to C_{p-1}.*
3. *[9, Proposition 4.4.17(3)] The automorphism group of $C_p \times C_p \times \cdots C_p$ (n factors) is $GL_n(p)$.*
4. *[9, Proposition 4.4.17(5)] The automorphism group of D_8 is isomorphic to D_8 and Aut $Q_8 \cong S_4$.*
5. *[9, Proposition 4.4.17(4)] For $n \neq 2, 6$ the automorphism group of S_n is isomorphic to S_n. For $n = 6$, the inner automorphism group is isomorphic to S_6 but has index 2 in Aut S_6.*
6. *[27, Theorem 5.7]The automorphism group of A_n is S_n for $n \neq 2, 6$.*

The last part of Theorem 1.1.4 is a slightly deeper result than the other statements, but is discussed in most graduate level permutation group courses. The proof of part (5) of Theorem 1.1.4 occurs in Chapter 3.1, although readers can also see it in the exercises of [9, Section 4.4].

For some groups, though, we still do not have a full picture of the automorphism group nor how individual automorphisms are constructed. In the next section we will discuss constructing automorphisms for semidirect products using information about the components of the product.

1.2 Constructing New Automorphisms From Old

In the case of a direct product $G_1 \times G_2$, it has long been known that an automorphism of either component can be extended to an automorphism of the direct product.

Theorem 1.2.1. *Let G_1 and G_2 be groups. If $\varphi \in \mathit{Aut}\, G_1$ then the extension of φ, denoted φ^*, to $G_1 \times G_2$ defined by*

$$\varphi^*(g_1, g_2) = (\varphi(g_1), g_2)$$

for all $g_1 \in G_1$ and $g_2 \in G_2$ is an automorphism of $G_1 \times G_2$.

Proof. It's clear that since φ is a bijection that φ^* is also a bijection. We proceed to prove that φ^* is a homomorphism.

Let $(g_1, g_2), (h_1, h_2)$ be elements of $G_1 \times G_2$. Since multiplication of the elements of $G_1 \times G_2$ is performed component-wise and φ is an automorphism, we make the following calculation.

$$\begin{aligned} \varphi^*(g_1h_1, g_2h_2) &= (\varphi(g_1h_1), g_2h_2) \\ &= (\varphi(g_1)\varphi(h_1), g_2h_2) \\ &= (\varphi(g_1), g_2)(\varphi(h_1), h_2) \\ &= \varphi^*(g_1, g_2)\varphi^*(h_1, h_2) \end{aligned}$$

Since φ^* is a homomorphism, we actually have that $\varphi^* \in \text{Aut } G_1 \times G_2$, as desired. ∎

The proof of this theorem can easily be generalized to the following.

Theorem 1.2.2. *Let G_1, G_2 be groups with $\varphi_i \in Aut\, G_i$ for $i = 1, 2$. Then $\varphi^* : G_1 \times G_2 \to G_1 \times G_2$ defined by $\varphi^*(g_1, g_2) = (\varphi_1(g_1), \varphi_2(g_2))$ is an automorphism.*

The situation becomes more complicated when working with semidirect products since the components do not commute. Suppose that $G = N \rtimes H$ is a semidirect product, and let $\sigma : H \to H$ and $\psi : N \to N$ be automorphisms of the two components of G. Then let φ be a function on G defined as follows:

$$\begin{aligned} &\varphi : G \to G \\ &\varphi(nh) = \psi(n)\sigma(h) \qquad \text{for all } n \in N \text{ and } h \in H. \end{aligned}$$

The function φ extends both σ and ψ but the question of interest is to determine when φ defines an automorphism of G. Since φ is a bijection we need only check to see when φ is a homomorphism.

Let n_1h_1, n_2h_2 be elements in G with $n_i \in N$, $h_i \in H$ for each i. Multiplication of these elements gives

$$n_1h_1 \cdot n_2h_2 = n_1 n_2^{h_1^{-1}} h_1h_2,$$

implying that φ will be a homomorphism if and only if

$$\psi(n_2^{h_1^{-1}}) = \psi(n_2)^{\sigma(h_1^{-1})}$$

holds for all $n_2 \in N$ and $h_1 \in H$. Thus φ is an automorphism of G if and only if

$$\psi(n^h) = \psi(n)^{\sigma(h)} \tag{1.1}$$

for all $n \in N$ and $h \in H$.

Note that if Equation 1.1 holds for all $n \in N$ and for two elements of H then it will also hold for the product of those two elements. Thus it suffices to check Equation 1.1 on the generators of H. Furthermore for each such generator, since both sides of Equation 1.1 are multiplicative in n, we need only check the equation on generators of N. In other words, φ is an automorphism of G if and only if Equation 1.1 holds for all generators of $n \in N$ and $h \in H$.

We will give examples of automorphisms successfully constructed in this manner in the later chapters. Typically, though, careful choice of ψ and σ is needed to satisfy Equation 1.1 and result in an automorphism.

Of particular interest to us in Chapter 2 are the groups in which $H \cong D_8$. Here we will use the presentation

$$D_8 = \langle x, t \mid x^2 = t^2 = (xt)^4 = 1 \rangle$$

where $z = (xt)^2$ is the unique non-identity central element of this group. Proposition 1.2.4, below, concludes that if N is abelian then for any automorphism of D_8 there is an automorphism ψ of N that satisfies Equation 1.1. In the proof of the proposition we use the Fundamental Theorem of Finitely Generated Abelian Groups [22, Hauptsatz I.13.12], which is stated first.

Theorem 1.2.3 (Fundamental Theorem of Finitely Generated Abelian Groups [22])**.** *Let A be a finitely generated abelian group. Then [there exist $n \in \mathbb{N}$ and $a_i \in A$ for $1 \leq i \leq n$ such that] $A = \langle a_1 \rangle \times \cdots \langle a_n \rangle$. There exists $k \in \mathbb{N}$ where for all i with $1 \leq i \leq k \leq n$ the element a_i has order $p_i^{\alpha_i}$ [for some positive integer α_i] and for all i with $k+1 \leq i \leq k+r = n$ the element a_i has infinite order. The number r and the prime powers $p_i^{\alpha_i}$ are unique to A. The number r is called the rank of A, denoted $r(A)$, and the values $p_i^{\alpha_i}$ are called the invariants, or elementary divisors, of A. Clearly these values uniquely describe a group up to isomorphism. We call $(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}, \infty, \cdots, \infty)$ the type of A. The set $\{a_i\}_{i=1}^n$ we call a basis of A.*

Proposition 1.2.4. *Let $W = N \rtimes D_8$ with N an abelian group of odd order and let D_8 be as described above. Assume that $n^z = n^{-1}$ for all $n \in N$.*

1. *If $N_1 = \{n \in N \mid n^x = n\}$ and $N_2 = \{n \in N \mid n^x = n^{-1}\}$, then $N = N_1 \times N_2$ is the direct product of these two subgroups with $N_1^t = N_2$. Thus $N_1 \cong N_2$ as abelian groups.*

2. *Any automorphism σ of D_8 extends to an automorphism of W.*

Proof. Let us denote D_8 as D for the duration of this proof. Note that $\langle x, z \rangle = \{1, x, z, xz = y\}$ is a normal elementary abelian subgroup of D of order 4, with $y = xz = x^t$.

Notice that since N is an abelian group of odd order and x has order 2, with some checking it follows that $N = N_1 \times N_2$ with subgroups N_1 and N_2 as described in the proposition (alternatively apply Fitting's Lemma [13, Theorem 5.2.3] for each Sylow subgroup of N). Clearly N_1 is the centralizer in N of x. Furthermore, since z inverts all elements of N and $y = xz = zx$, we see that N_2 is the centralizer in N of y. Yet $x^t = y$, so

$$N_1^t = (C_N(x))^t = C_N(x^t) = C_N(y) = N_2.$$

In particular, $N_1 \cong N_2$ as abelian groups.

For part (2), given any automorphism σ of D we need to construct an appropriate automorphism ψ of N so that Equation 1.1 is satisfied. By the above we have that $N = N_1 \times N_2$ with $N_1^t = N_2$.

Theorem 1.2.3 promises that if $N = N_3 \times N_4$ where $N_3 \cong N_4$ then $N_3 \cong N_1$, as well. We will then construct subgroups N_3 and N_4 to help us define an automorphism ψ of N that satisfies Equation 1.1.

First notice that

$$D = \langle \sigma(x), \sigma(t) \mid \sigma(x)^2 = \sigma(t)^2 = \sigma(xt)^4 = 1 \rangle.$$

Since z is the unique non-identity central element of D we have that $\sigma(z) = z$. Thus $\sigma(z)$ also acts by inversion on N. A second application of part (1) implies that $N = N_3 \times N_4$ where

$$N_3 = \{n \in N \mid n^{\sigma(x)} = n\} \qquad \text{and} \qquad N_4 = \{n \in N \mid n^{\sigma(x)} = n^{-1}\}.$$

Additionally we have that $N_3^{\sigma(t)} = N_4$.

As N_1 and N_3 are isomorphic, we can let $\psi_1 : N_1 \to N_3$ be any isomorphism between them. Using $N_1^t = N_2$ and $N_3^{\sigma(t)} = N_4$, we can define a function ψ_2 as given.

$$\psi_2 : N_2 \to N_4$$
$$\psi_2(n^t) = \psi_1(n)^{\sigma(t)}$$

Since both ψ_1 and conjugation by $\sigma(t)$ are injective homomorphisms, ψ_2 can easily be shown to be an isomorphism from N_2 to N_4.

Given $n \in N$ there are unique $a, b \in N_1$ such that $n = ab^t$. This allows us to combine ψ_1 and ψ_2 to define an automorphism ψ on N.

$$\psi(n) = \psi(ab^t) = \psi_1(a)\psi_2(b^t) = \psi_1(a)\psi_1(b)^{\sigma(t)}$$

Now extend $\psi : N \to N$ and $\sigma : D \to D$ to an automorphism of W using Equation 1.1. One need only check that Equation 1.1 holds for an element of N_1 or N_2 and for the generators of D. Thus each automorphism of D extends to an automorphism of W. ■

The example below illustrates one way to extend an automorphism of D_8 to a larger group with the structure required in the preceeding result.

Example 1.2.5. Let W be a Sylow-2 subgroup of S_8, namely

$$W = \langle (1\ 2), (3\ 4), (5\ 6), (7\ 8), (1\ 5)(2\ 6), (1\ 3)(2\ 4)(5\ 7)(6\ 8) \rangle.$$

Notice that the four transpositions $(a\ a+1)$ given in W generate a elementary abelian normal 2-subgroup of order 2^4, which we will call B. If we let $x = (1\ 5)(2\ 6)$ and $t = (1\ 3)(2\ 4)(5\ 7)(6\ 8)$ then $\langle x, t\rangle$ is isomorphic to D_8. Moreover it's easy to show that $W = B \rtimes \langle x, t\rangle$.

The automorphism of $\langle x, t\rangle$ defined by $x \mapsto t^x$ and $t \mapsto x$ can be extended to be an automorphism of W. One such extension would map the generators of W as given below.

$$\begin{aligned}
(1\ 2) &\mapsto (1\ 4)(2\ 3)(5\ 7)(6\ 8)\\
(3\ 4) &\mapsto (1\ 7)(2\ 8)(3\ 6)(4\ 5)\\
(5\ 6) &\mapsto (1\ 3)(2\ 4)(5\ 8)(6\ 7)\\
(7\ 8) &\mapsto (1\ 8)(2\ 7)(3\ 5)(4\ 6)\\
(1\ 5)(2\ 6) &\mapsto (1\ 7)(2\ 8)(3\ 5)(4\ 6)\\
(1\ 3)(2\ 4)(5\ 7)(6\ 8) &\mapsto (1\ 5)(2\ 6)
\end{aligned}$$

Let b_i be the i^{th} transposition generating W for $i = 1, 2, 3, 4$ then a formula for how to calculate the outputs of this isomorphism is

$$b_i \mapsto b_i b_j t^{x^k}$$

for $i = 1, 2, 3, 4$ where $j \equiv i + 1$ modulo 4 and $k \equiv i + 1$ modulo 2. It's straightforward, though tedious, to check that this assignment does in fact define an automorphism of W.

The study of how automorphisms of a semidirect product can be induced from the components is a complex area of research. A deeper introduction to the topic can be found in [6].

1.3 Cohomology

In this section we relate the automorphisms of a normal subgroup to those of the larger group from a different angle. Our examinations in this section do not require that the group G be a semidirect product but we rely on results from cohomology. The relation between the automorphism group of a normal subgroup and that of a larger group has long been studied, with the next proposition giving what might easily be the inspiration for these studies.

Proposition 1.3.1 (Proposition 4.4.13 [9])**.** *Let N be a normal subgroup of a group G. Then G acts by conjugation on N as automorphisms of N. More specifically, the action of G on N by conjugation is defined for each $g \in G$ by*

$$n \mapsto n^g \qquad \textit{for each } n \in N.$$

For each $g \in G$, conjugation by g is an automorphism of N. The permutation representation afforded by this action is a homomorphism of G into Aut N with kernel $C_G(N)$. In particular, $G/C_G(N)$ is isomorphic to a subgroup of Aut N.

Proposition 1.3.1 shows how to restrict the inner automorphisms of the larger group G to a normal subgroup N. The next proposition shows that information about how an automorphism behaves on a normal subgroup can dictate its behavior on the larger group.

Proposition 1.3.2. *Let N be a normal subgroup of a group G where $C_G(N) = 1$. If φ is an automorphism of G that fixes the elements of N point-wise then φ is the identity automorphism.*

Proof. Let φ be an automorphism of G fixing each element of N. Consider $g \in G$ and $n \in N$. The element n^g belongs to N and thus is fixed by φ. The result of applying conjugating n by $\varphi(g)$ follows.

$$\begin{aligned}
n^{\varphi(g)} &= \varphi(g)^{-1} n \varphi(g) \\
&= \varphi(g^{-1}) n \varphi(g) \\
&= \varphi(g^{-1} n g) \\
&= \varphi(n^g) \\
&= n^g
\end{aligned}$$

The above is true for all $n \in N$, hence $\varphi(g)g^{-1}$ centralizes N. Yet the centralizer of N is trivial, leading to the conclusion that $\varphi(g) = g$ for all $g \in G$. ■

Proposition 1.3.2 is a well-known fact to those who study cohomology of groups. The hypothesis that $C_G(N) = 1$ is quite strong; to relax that requirement we introduce cohomology. However, the next theorem from Burnside [4] might well be the inspiration for the connection between automorphisms and cohomology. Note Burnside states the theorem with the conclusion that $|\varphi|$ divides $|N|$, however the proof is simplified by alteration we give here.

Theorem 1.3.3 (Section 73 Theorem VII [4]). *Let $N \trianglelefteq G$. If $\varphi \in Aut\ G$ such that φ fixes both N and G/N element-wise then $|\varphi|$ divides $|N|$.*

Proof. Let $x \in G - N$. Since $(xN)^{\varphi} = x^{\varphi}N = xN$ we know that there exists $n \in N$ such that $n^{\varphi} = xn$. Moreover $n^{\varphi} = n$. This yields:

$$x^{\varphi^a} = (x^{\varphi})^{\varphi^{a-1}} = (xn)^{\varphi^{a-1}} = (x^{\varphi}n)^{\varphi^{a-2}} = \cdots = xn^a$$

for all $a \in \mathbb{N}$. Thus if $|n| = b$ we know $x^{\varphi^b} = x$. If we consider $x \in N$ then $x^{\varphi} = n$, so for all $g \in G$ there exists b such that $b \bigm| |N|$ and $x^{\varphi^b} = x$. Therefore by Lagrange's Theorem [9, Theorem 3.2.8] the order of φ must divide the order of N. ■

Let G be a group with $N \trianglelefteq G$. The automorphisms of G that fix both N and G/N element-wise give a connection from group theory into cohomology theory, as we see in the next theorem. Recall that G acts on N by conjugation, since $N \trianglelefteq G$. That action can be extended to an action of G/N on N. Since $\mathrm{Z}(N)$ is also normal in G, and now abelian, this group is a G/N-module under this action. We will denote the first cohomology group of G/N with coefficients in $\mathrm{Z}(N)$ by $H^1(G/N, \mathrm{Z}(N))$. We remind readers that the latter group is defined to be the quotient of the group of 1-cocycles (denoted $Z^1(G/N, \mathrm{Z}(N))$) by the group of 1-coboundaries (denoted $B^1(G/N, \mathrm{Z}(N))$). This idea is generalized to the n^{th} cohomology group of G/N with coefficients in $\mathrm{Z}(N)$, denoted $H^n(G/N, \mathrm{Z}(N))$ for any positive integer n.

Theorem 1.3.4 (Satz I.17.1 [22]). *Suppose G is a group and let $N \trianglelefteq G$.*

1. *If φ is an automorphism of G that fixes both N and G/N element-wise then $F(gN) = g^{-1}\varphi(g)$ is a 1-cocycle.*

2. *Let* $F \in Z^1(G/N, Z(N))$. *Then the map* $\varphi : G \to G$ *defined by* $\varphi(g) = gF(gN)$ *is an automorphism of* G *that fixes* N *and* G/N *element-wise.*

3. *Let* φ *be an automorphism of* G *that fixes the elements of* N *and* G/N *point-wise. Let* $F \in Z^1(G/N, Z(N))$ *be the corresponding 1-cocycle given in (a). Then* $F \in B^1(G/N, Z(N))$ *if and only if there exists* $x \in N$ *with* $\varphi(g) = g^x$ *for all* $g \in G$.

Corollary 1.3.5. *Let* G *be a group with* $N \trianglelefteq G$. *Suppose that* A *is the subgroup of all automorphisms of* G *that fix the elements of* N *and* G/N *point-wise. Then the map* $\Phi : A \to H^1(G/N, Z(N))$ *defined by* $\varphi \mapsto F$, *where* F *is described in Theorem 1.3.4(1), is surjective and the kernel is* $A \cap \mathit{Inn}\, G$. *Moreover, if* $H^1(G/N, Z(N)) = 1$ *then* $A \leq \mathit{Inn}\, G$.

The corollary summarizes the main effect of Theorem 1.3.4 upon which we rely. Proposition 1.3.6, which follows, demonstrates how we will make use of the relationship between $H^1(G/N, \mathrm{Z}(N))$ and Aut G.

Proposition 1.3.6. *Let* G *be a group and suppose there exists a normal subgroup* N *of* G *with* $C_G(N) \leq N$ *and* $H^1(G/N, Z(N)) = 1$. *Then any automorphism of* G *that fixes* N *element-wise is an inner automorphism of* G *and induces the identity on* G/N.

Proof. Suppose that φ is an automorphism of G that fixes each element of N. Let $n \in N$ and $g \in G$. Since N is normal and φ is trivial on N the following equalities hold.

$$\begin{aligned} n^{\varphi(g)} &= \varphi(n^g) \\ &= n^g \end{aligned}$$

Therefore $\varphi(g)g^{-1}$ centralizes n all $n \in N$. As $C_G(N) \leq N$ there must exist $n_1 \in N$ such that

$$\varphi(g) = n_1 g \qquad \text{for each } g \in G.$$

Modulo N, the automorphism φ fixes the elements of G. Hence φ fixes both N and G/N element-wise. By Theorem 1.3.4(c) the automorphism corresponds to an element of $H^1(G/N, \mathrm{Z}(N))$. This latter group is trivial, so φ and the identity automorphism of G have the same image modulo Inn G. In other words, φ is inner on G. ∎

Proposition 1.3.6 is also a fact commonly known to those who study the cohomology of groups. We see from Proposition 1.3.6 how important it is to understand the first cohomology group associated with a given normal subgroup N and how most of our calculations will require a trivial first cohomology group. Fortunately we are often able to rely on the following facts.

Lemma 1.3.7 (Corollary 17.2.29 [9]). *Suppose G is a finite group whose order is relatively prime to the exponent of the G-module A. Then $H^m(G, A) = 1$ for all $m \geq 1$. In particular, if A is a finite G-module with $(|G|, |A|) = 1$ then $H^m(G, A) = 1$ for all $m \geq 1$.*

Lemma 1.3.8. *If G is a finite group with $N \trianglelefteq G$ and $Z(N) = 1$ then $H^m(G/N, Z(N)) = 1$ for all $m \geq 1$.*

The last lemma is of course trivial, but we shall use this fact in Section 3.3 when discussing complete wreath products. Also in that section we will rely on the proposition by Schmid [31], below, which brings the second cohomology group into the picture. Fortunately in our case, the first cohomology group will be trivial and this in turn implies that the second cohomology group is also trivial.

Recall that if $N \trianglelefteq G$ then for any $g \in G$ the automorphism τ_g, fixes N set-wise. So $\tau_{g|N} \in \text{Aut } N$ for all $g \in G$. Of course, if $g \in C_G(N)$ then $\tau_{g|N}$ is the identity automorphism of N.

Proposition 1.3.9 (Proposition 2.1 [31]). *Suppose that N is a normal subgroup of G such that $C_G(N) = Z(N)$ and $H^2(G/N, Z(N)) = 1$. Let D denote the group of automorphisms of N induced by G and let A_N be any automorphism group of N. Then there is an automorphism group A of G normalizing N and inducing A_N if and only if A_N normalizes D.*

We can restate Proposition 1.3.9 in a form that will be more useful to us in the later chapters.

Proposition 1.3.10. *Let G be a group with $N \trianglelefteq G$ such that $C_G(N) \leq N$ and $H^2(G/N, Z(N)) = 1$. Let $D = \{\tau_{g|N} \mid g \in G\}$ and let $A \leq Aut\, N$. Then there is an $A^* \leq Aut\, G$ which fixes N set-wise and restricts to A if and only if $A \leq N_{Aut\, N}(D)$.*

We will use Proposition 1.3.10 to force automorphism subgroups to be self-normalizing, and conversely the proposition allows us to work with restrictions of automorphism subgroups.

For readers who desire to learn more about cohomology of groups we recommend the discussions found in Chapter 17 Section 2 of [9] or Chapter I Sections 16-17 of [22]. Additionally, [3] provides a broader viewpoint of cohomology.

Chapter 2

Wreath Products

In this chapter we show that the base group of a finite permutational wreath product $G \wr H$ is not characteristic precisely when $G = A \rtimes C_2$, where A is an abelian subgroup of odd-order and C_2 acts on A by inversion. First we will lay the groundwork in Section 2.1. In Section 2.2 we prove the main theorem and explore its implications in a few specific cases. In particular we confirm previous results by Y. V. Bodnarchuk [2] in the case that H is acting transitively. In the last section we compute the order of the automorphism group of $G \wr C_2$ when the base group is not characteristic and indicate some of the structure.

Throughout this chapter we very carefully reserve the term "transposition" to refer exclusively to a permutation that moves exactly two elements. An "involution" will be any element with order less than or equal to 2. Notice that a non-trivial involution is an element of order 2 but need not be a transposition.

The results for this chapter were achieved in collaboration with Benjamin Brewster (dissertation advisor of the author) and Donald Passman, from the University of Wisconsin at Madison.

2.1 Background

The notation that we set in this section will be maintained throughout the chapter and wherever wreath products are discussed in this dissertation. Let G be a non-trivial finite group and H be a finite permutation group of degree n acting faithfully on a set Ω, hence $H \leq S_\Omega$. The permutational wreath product of G with H is defined as follows.

Definition 2.1.1. The *permutational wreath product* of G with H, denoted $G \wr H$, is the semidirect product $B \rtimes H$ where $B = \{f : \Omega \to G\}$. In this semidirect product multiplication is defined as $(f_1h_1)(f_2h_2) = f_1f_2^{h_1^{-1}}h_1h_2$ where conjugation of $f \in B$ by $h \in H$ is given by $f^h(\omega) = f(\omega h^{-1})$.

To be clear we remind readers that multiplication of elements of B is done point-wise. Given the definition, it's easy to determine the order of $G \wr H$.

Proposition 2.1.2. *Let G be a finite group and H a finite permutation group of degree n. Then $|G \wr H| = |G|^n|H|$.*

To describe other characteristics of the wreath product, though, we need to introduce more terminology and notation. Let $W = G \wr H$ denote a permutational wreath product of G with H. Typically we drop the word "permutational" and call W simply the wreath product of G with H. We refer to B as the *base group* of W and identify H with the subgroup of W formed by elements of the form fh where $f(\omega) = 1$ for all $\omega \in \Omega$. This group will be referred to as the *top group* of W, whereas G will be referred to as the *bottom group.*

Since W is a semidirect product each element of W can be written uniquely as bh where $b \in B$ and $h \in H$. Those elements bh where $h \neq 1$ are not elements of B and hence will be referred to as *non-base* elements. Elements of B that are trivial for all but at most one coordinate shall be referred to as *coordinate elements* of B. For $\omega \in \Omega$ a *coordinate element of type* ω will refer to coordinate elements that are trivial for all inputs except possibly ω. The subgroup of all coordinate elements of a fixed type ω will be denoted by G_ω. Notice that $G_\omega \cong G$ and further that $G_\omega \trianglelefteq B$, hence is subnormal in W.

If one regards the elements of B as n-tuples of elements of G then under our notation the action of h^{-1} on the indices of the components of the n-tuple b is equivalent to the action of h on b. And so we have for $b \in B$ and $h \in H$:

$$b^h = (g_\omega)^h_{\omega \in \Omega} = (g_{\omega h^{-1}})_{\omega \in \Omega}.$$

As B is a direct product, it is natural to discuss the *diagonal of* B as the subgroup of all n-tuples with the same coordinate values, or

$$D(B) = \{f \in B \mid \ \exists g \in G \text{ for which } f(\omega) = g \text{ for all } \omega \in \Omega\}.$$

We call $D(B)$ the *diagonal of* W, as well.

With some notation established, we pause to note that this definition of wreath product places no requirements on the action of H other than faithfulness. If H is acting on itself by the right (or left) regular representation then $G \wr H$ is called the *standard wreath product* of G with H. Standard wreath products were studied in depth by P. Neumann in his 1964 article [26]. We will examine standard wreath products again in Section 3.3.3. Another common situation is to require H to be transitive. We will make this assumption in Subsection 2.2.2.

In order to discuss wreath products we introduce a few preliminary results, many of which are detailed in references such as [22] and [24]. We begin with two notes about coordinate elements.

Proposition 2.1.3. *For a fixed* $\omega \in \Omega$, *the set of all coordinate elements of type* ω *generate* G_ω. *The set of all coordinate elements generates* B.

Proposition 2.1.4. *Fix* $\omega \in \Omega$ *and* $h \in H$. *If* f *is a coordinate element of type* ω *and* $\omega h = \alpha$ *for some* $\alpha \in \Omega$, *then* f^h *is a coordinate element of type* α. *Hence* $G_\omega^h = G_\alpha$. *The set of all coordinate elements is fixed set-wise by the action of* H *on* B.

Propositions 2.1.3 and 2.1.4 are fairly obvious, given the definitions of coordinate elements and G_ω. It is important, however, to keep in mind that the elements of the base are products of coordinate elements of different types.

Proposition 2.1.5. *Let* $W = G \wr H$ *where* G, H *are finite groups and* H *acts of degree* n *on* Ω. *Then:*

1. *The centralizer of* B *in* W *is* $Z(B)$.
2. *The centralizer of* H *in* B *is the subgroup of all functions whose value is constant on the orbits of* H. *In particular the diagonal of* W *centralizes* H.
3. *The center of* W *is the the subgroup of all functions of* B *that are constant on the orbits of* H *and take values in* $Z(G)$.

Proof. Since H is a permutation group, it acts faithfully on Ω. Hence every $h \in H$ moves at least one point in Ω, implying that every $h \in H$ moves at least one G_ω for $\omega \in \Omega$. Therefore no element of H centralizes B. Moreover W/B acts as H does, hence the non-base elements of W cannot centralize B. Therefore $C_W(B) \leq B$ and hence $C_W(B) = \mathrm{Z}(B)$.

Now suppose $b \in C_B(H)$ and let $h \in H$ such that $\alpha h^{-1} = \omega$. Then we know that $b^h = b$ and it follows that

$$b(\alpha) = b^h(\alpha) = b(\alpha h^{-1}) = b(\omega).$$

Hence b must be constant on the orbits of H. Therefore $C_B(H)$ is equal to the set of all functions $b \in B$ whose value is constant on the orbits of H. Observe that for all $f \in D(B)$, the value of f is constant on all $\omega \in \Omega$ so certainly $f \in C_B(H)$. Thus $D(B) \leq C_B(H)$.

Lastly, the center of W is a subgroup of $C_W(B) = \mathrm{Z}(B)$. Yet these elements must also commute with H, so $\mathrm{Z}(W) \leq C_B(H) \cap \mathrm{Z}(B)$. Let us call this intersection C, and notice that

$$C = \{b \in B \mid b(\omega) = b(\omega h^{-1}) \in \mathrm{Z}(G) \text{ for all } \omega \in \Omega \text{ and } h \in H\}.$$

However all elements of C commute both with the elements in B and those in H. Since W is a semidirect product, all elements of W can be written (uniquely) as a product of an element in B with one from H. Therefore $c \in C$ commutes with every element in W and hence $C = \mathrm{Z}(W)$. ■

We further investigate the structure of centralizers in Section 2.2.1, where we consider the centralizer of an element in the base.

We may also view W as a permutation group. If G is a permutation group acting on a set Δ then W acts on $\Delta \times \Omega$ by

$$(\delta, \omega)bh = (\delta b(\omega), \omega h).$$

It is not difficult to check that this in fact defines an action of W; we leave the checking to the reader. If K is a finite permutation group acting on a set Λ, then both $(G \wr H) \wr K$ and $G \wr (H \wr K)$ act on the set $\Delta \times \Omega \times \Lambda$. In fact, we have:

Theorem 2.1.6 (Hilfsatz I.15.4 [22]). *Let G, H, K be permutation groups of the sets Δ, Ω, Λ respectively. Then the permutation groups $(G \wr H) \wr K$ and $G \wr (H \wr K)$ are equal, so we naturally identify their actions on $(\Delta \times \Omega) \times \Lambda$ and $\Delta \times (\Omega \times \Lambda)$.*

The wreath product construction of groups is a rich source of examples of groups, and provides insight into general group structure. The first example of a wreath product that one comes across is D_8, the dihedral group of order 8.

Proposition 2.1.7. *The dihedral group of order 8 is isomorphic to the standard wreath product of C_2 with C_2.*

Proof. Using the same notation as in Chapter 1, we know that x is a generator of D_8 with order 2 and z is the unique non-identity central element. Then xz is another element of order 2 in D_8 that commutes with x. Let $B = \langle x \rangle \times \langle xz \rangle$. The element t interchanges x and xz under conjugation, hence $D_8 \cong C_2 \wr C_2$. ■

Less trivial examples of wreath products lie within the symmetric groups. Let p be a prime and m be an integer greater than 1. It is well-known that a Sylow p-subgroup of S_{p^m} is an iterated wreath product.

Proposition 2.1.8 (Section 2 [27]). *Let p be a prime and m a positive integer. The Sylow p-subgroup of S_{p^m} is isomorphic to $C_p \wr \cdots \wr C_p$ (m factors).*

Of course, if $m = 1$ it's easy to see that the Sylow p-subgroups of S_p are isomorphic to C_p. For $m \geq 2$, the proof relies on partitioning the p^m points into sets of size p and using induction to prove that $C_p \wr \cdots \wr C_p$ acts on the p^m points by permuting both the elements in each set and the sets themselves. Then the wreath product acts faithfully of degree p^m and hence must be isomorphic to a subgroup of S_{p^m}. Yet the order of this group forces its image to be a Sylow p-subgroup of S_{p^m}.

Aside from providing a source of interesting examples of groups, and permutation groups in particular, the wreath product construction is important to the study of groups from a more general standpoint. Kaluzhnin and Krasner [24, Theorem 2.6] proved a version of Theorem 2.1.9 in the 1950's and B. H. Neumann proved another version [22, Satz I.15.12]. Here we present the statement from [7] whose notation fits in nicely with our own.

Theorem 2.1.9 (Universal Embedding Theorem). *[7, Theorem 2.6A] Let $G \trianglelefteq K$. Let K/G act on itself by the right regular action and form $G \wr (K/G)$ from this action. There is an injective homomorphism $\varphi : K \to G \wr (K/G)$ that maps G into the base of $G \wr (K/G)$.*

Proof. Let $H = K/G$ and let $\psi : K \to H$ be the quotient homomorphism with kernel G. Let $T = \{t_h \mid h \in H\}$ be a complete set of representatives for the right cosets of G in K where $\psi(t_h) = h$. First we observe a few facts about ψ.

If $k \in K$ then

$$\psi(t_h k) = \psi(t_h)\psi(k) = h\psi(k).$$

Then notice that

$$\begin{aligned} \psi(t_h k t^{-1}_{h\psi(k)}) &= \psi(t_h k)\psi(t^{-1}_{h\psi(k)}) \\ &= h\psi(k)(h\psi(k))^{-1} \\ &= 1. \end{aligned}$$

Hence for all $h \in H$ we have that $t_h k t_{h\psi(k)}$ is in the kernel of ψ, meaning that $t_h k t_{h\psi(k)} \in G$.

For $k \in K$ let us define a function $f_k : H \to G$ by $f_k(h) = t_h k t^{-1}_{h\psi(k)}$. Notice that

$$f_{km}(h) = t_h k m t^{-1}_{h\psi(km)}.$$

Consider $f_k f_m^{\psi(k)^{-1}}(h)$, for reasons that will soon become clear.

$$\begin{aligned} f_k f_m^{\psi(k)^{-1}}(h) &= f_k(h) f_m^{\psi(k)^{-1}}(h) \\ &= f_k(h) f_m(h\psi(k)) \\ &= t_h k t^{-1}_{h\psi(k)} t_{h\psi(k)} m t^{-1}_{h\psi(k)\psi(m)} \\ &= t_h k m t^{-1}_{h\psi(km)} \\ &= f_{km}(h) \end{aligned}$$

The above holds for all $h \in H$, hence $f_k f_m^{\psi(k)^{-1}} = f_{km}$.

Now we are ready to define the homomorphism $\varphi : K \to G \wr H$, by sending $k \in K$ to $f_k \psi(k)$. Notice that $\psi(k) \in H$ and f_k is a function with domain H and codomain G. Therefore $\varphi(k)$ is an element of $G \wr H$. To show that φ is a homomorphism, let $k, m \in K$. Then notice that

$$\begin{aligned} \varphi(k)\varphi(m) &= f_k\psi(k) f_m \psi(m) \\ &= f_k f_m^{\psi(k)^{-1}} \psi(k)\psi(m) \\ &= f_{km}\psi(km) \\ &= \varphi(km), \end{aligned}$$

using the facts just established and the knowledge that ψ is a homomorphism. Since $G \wr H$ is a wreath product, the kernel of φ is all elements $k \in K$ for which f_k is the trivial function and $\psi(k) = 1$. So on one hand, $f_k(1) = 1$ but by its definition

$$f_k(1) = t_1 k t^{-1}_{1\psi(k)}.$$

Thus

$$\begin{aligned} k &= t_1 f_k(1) t^{-1}_{1\psi(k)} \\ &= t_1 1 t_1^{-1} && \text{since } \psi(k) = 1 \\ &= t_1 t_1^{-1} \\ &= 1. \end{aligned}$$

Hence φ is an injective homomorphism, as desired.

If $k \in G$ then $\varphi(k) = f_k \psi(k) = f_k$ since G equals the kernel of ψ. Hence $\varphi(k)$ is an element of the base, as desired. ■

Notice also that φ as defined in the proof sends $kG \in K/G$ to $f\ kG$ where $f : H \to G$ is the identity function and if $h = kG$ then $\varphi(h) = \varphi(kG) = k^{\varphi}G$. Then for $h \in H$ we have

$$\begin{aligned} f_h(h_1) &= t_{h_1} h t^{-1}_{h_1\psi(h)} \\ &= h_1 h (h_1 h)^{-1} \\ &= 1 \end{aligned}$$

for all inputs $h_1 \in H$. Therefore $\varphi(h) = f_h h = f/h$ where f is the identity-valued function.

This theorem tells us that any finite group can be embedded in a wreath product. Just as we study symmetric groups because any group can be embedded in a symmetric group, we similarly study wreath products. In fact, if G is a group with a chain of normal subgroups then repeated use of Theorem 2.1.9 allows for the embedding of G in an iterated wreath product. This actually allows for one to embed any finite group in an iterated wreath product of simple groups, and in particular a solvable group can be embedded in an iterated wreath product whose components have prime order.

The next section will address the question of when the base of a wreath product is characteristic. For additional information about wreath products, readers may refer to [22], [24], or [27].

2.2 The Base of a Permutational Wreath Product

In 1964 P. Neumann determined precisely when the base B is characteristic in W for the standard wreath product case [26]. In order to present this characterization, we first give a definition.

Definition 2.2.1 (Definition 9.2 [26])**.** A group G is a *dihedral group* if $G = A \rtimes \langle x \rangle$ where A is a (possibly trivial) abelian group on which x by inversion and $\langle x \rangle \cong C_2$. A *special dihedral group* is a dihedral group in whose abelian normal subgroup A there is a unique square root for each element.

Theorem 2.2.2 (Theorem 9.1 [26])**.** *Let G, H be groups with H acting on itself via the regular representation. Then the standard wreath product $G \wr H$ has a characteristic base if and only if $H \neq C_2$ or G is not special dihedral.*

In 1984, Y. V. Bodnarchuk extended this result to the case where H acts transitively on Ω. Bodnarchuk's final result is:

Theorem 2.2.3 (Theorem 1 [2])**.** *Let G and H be groups where H acts transitively on a set Ω of cardinality n. If the base of the wreath product $G \wr H$ is not characteristic then G is a special dihedral group and $H = C_2 \wr K$ where K is a group acting with degree $n/2$.*

Both results include the case of an infinite wreath product. As our work is focused on finite groups, we introduce the following definition.

Definition 2.2.4. A finite dihedral group G is *odd-dihedral* if its abelian normal subgroup is of odd order.

Proposition 2.2.5. *Let G be a finite group. Then G is odd-dihedral if and only if G is special dihedral.*

Proof. Let G be a finite group. Suppose first that G is odd-dihedral with $G = A \rtimes \langle x \rangle$, where A is an odd-order abelian group and conjugation by x inverts the elements of A. The map $a \mapsto a^2$ is a homomorphism that, since A has odd-order, has a trivial kernel. Hence this map is injective and the non-identity elements of A have a unique square root.

Suppose instead that G is special dihedral so that G has a normal abelian subgroup A with index 2 whose complement acts by inversion and where the elements of A each have a unique square root. Let us assume that A has an element a with $a^{2k} = 1$ for a positive integer k. Then we know $(a^k)^2 = 1$ so, without loss of generality, suppose that $a^2 = 1$. Let $b \in A$ and suppose that c is the unique square root of b. As A is abelian we have

$$(ca)^2 = c^2a^2 = c^2 = b.$$

Due to uniqueness we have that $ca = c$ and hence $a = 1$. Therefore A does have elements of even order, meaning that $|A|$ is odd. Hence G is odd-dihedral. ■

Since the groups we discuss are finite, the term "odd-dihedral" is more descriptive of the structure of the group. In this section we will show that if the base of a permutational wreath product is not characteristic then G is odd-dihedral, as well as give some additional structure about the group H. Our approach differs from the approaches of Neumann and Bodnarchuk, and gives insight into the structure of the centralizer of an element in a wreath product. Additionally we will strengthen Bodarchuk's result to a biconditional statement.

2.2.1 Arbitrary Actions

Before investigating the action of automorphisms on a wreath product, we describe the structure of the centralizer of a non-base element. Ultimately we will count the number of conjugates with which an element of a wreath product commutes and connect that information to the structure of G.

Proposition 2.2.6. *Let fh be a non-base element of W where h has t orbits in its action on Ω. Let $\mathcal{O}$ an orbit of h with size r and $T = \{b \in B \mid b(\omega) = 1$ for all $\omega \notin \mathcal{O}\}$.*

1. *If $b \in C_T(fh)$ then the value of b on any element in $\mathcal{O}$ is uniquely determined by the value of b on a fixed $\alpha \in \mathcal{O}$, as follows:*

   $$b(\alpha h^m) = b(\alpha)^{f(\alpha)f(\alpha h)f(\alpha h^2)\cdots f(\alpha h^{m-1})} \qquad \text{for } 1 \leq m \leq r.$$

   Therefore α must be an element of $\mathcal{O}$ where $b(\alpha)$ centralizes the exponent

   $$f(\alpha)f(\alpha h)f(\alpha h^2)\cdots f(\alpha h^{r-1}).$$

Conversely, if $\alpha \in \mathcal{O}$ where $b(\alpha)$ centralizes the above value and is completely determined by its value on α then $b \in C_T(fh)$.

2. *The order of the centralizer in T of fh is less than or equal to $|G|$.*
3. *The order of $C_B(fh)$ is less than or equal to $|G|^t$.*

Proof. Let $b \in B$. Then b can commute with fh if and only if the following holds:

$$\begin{aligned} b(fh) = (fh)b \quad &\Leftrightarrow \quad f^{-1}bfh = hb \\ &\Leftrightarrow \quad b^f = b^{h^{-1}} \\ &\Leftrightarrow \quad b(\omega)^{f(\omega)} = b(\omega h) \text{ for all } \omega \in \mathcal{O}. \end{aligned}$$

Therefore the value of b on any element in $\mathcal{O}$ is determined by the value of b on one fixed point in $\mathcal{O}$. If we fix $\alpha \in \mathcal{O}$ this gives a formula:

$$b(\alpha h^m) = b(\alpha)^{f(\alpha)f(\alpha h)f(\alpha h^2)\cdots f(\alpha h^{m-1})} \qquad \text{for } 1 \leq m \leq r. \tag{2.1}$$

We then have

$$|C_T(fh)| \leq |G|.$$

Notice that $b(\alpha) = b(\alpha h^r)$ so the choice of $b(\alpha)$ is restricted to the elements of G that centralize $f(\alpha)f(\alpha h)f(\alpha h^2)\cdots f(\alpha h^{r-1})$. In particular, if G is non-abelian then $|C_G(fh)|$ may be strictly less than $|G|$.

On the other hand, if there exists an $\alpha \in \mathcal{O}$ for which $b(\alpha)$ commutes with $f(\alpha)f(\alpha h)f(\alpha h^2)\cdots f(\alpha h^r$ and b satisfies the Formula 2.1, then b must commute with fh. Hence an element of $C_G(fh)$ is uniquely determined by its values on fixed representatives for each orbit. Since there are t orbits, therefore $|C_B(fh)| \leq |G|^t$. ■

One can separtely prove the next proposition, which is a more general version of Proposition 2.2.6. However, the proof is very similar so we omit it. Remember that if $M \leq W$ then MB/B is isomorphic to a subgroup of H. So one can define an action of MB/B on Ω through this isomorhism and extend that to an action of M on Ω with kernel $M \cap B$.

Proposition 2.2.7. *Let $M \leq W$ such that $M \not\leq B$. Then $|C_B(M)| \leq |G|^t$ where t is the number of orbits of M acting on Ω.*

Throughout this section, for $x \in W$ let $\kappa(x)$ denote the number of W-conjugates of x that do not commute with x. In other words,

$$\kappa(x) = |\{x^w \in W \mid [x, x^w] \neq 1\}|.$$

The centralizer structure developed in Proposition 2.2.6 allows us to place a lower bound on κ for a non-base element of W. Before the proposition, recall that if $K \leq L$ where K, L are groups and $x \notin K$ then there is a one-to-one correspondence between the set of cosets of $C_K(x)$ and the set of commutators $[x, K] = \{[x, k] \mid k \in K\}$. The correspondence is defined by mapping $kC_K(x)$ to $[x, k]$, hence the map is onto. Less obvious is the fact that the map will be well-defined but straightforward calculations reveal that $kC_K(x) = mC_K(x)$ if and only if $[x, k] = [x, m]$. We will use this fact in the proof of the following proposition.

Proposition 2.2.8. *Let fh be a non-base element of W. Let $\mathcal{O}$ be one of t orbits of h in its action on Ω and suppose that $|\mathcal{O}| = r$. Let $T = \{b \in B \mid b(\omega) = 1$ for all $\omega \notin \mathcal{O}\}$. The set of commutators $[fh, T]$ has size at least as big as $|G|^{r-1}$. Furthermore,*

$$\kappa(fh) \geq |G|^{r-1} - |[fh, T] \cap C_T(fh)| \geq |G|^{r-1} - |G|.$$

Proof. There is a 1-1 correspondence between the cosets of $C_T(fh)$ in T and $[fh, T]$ as sets. Proposition 2.2.6 showed that $|C_T(fh)| \leq |G|$. Since $|T| = |G|^r$ we then have that $|[fh, T]| \geq |G|^{r-1}$.

To develop the lower bound on $\kappa(fh)$ we count only the number of T-conjugates of fh with which fh does not commute. The number of conjugates of fh *in T* is equal to the index of $C_T(fh)$, i.e., the number of T-conjugates of fh equals $|[fh, T]|$. Notice that if $v \in T$ we have $(fh)^v = (fh)[fh, v]$; thus conjugates fh and $(fh)^v$ commute if and only if fh and $[fh, v]$ commute. Since $T \trianglelefteq \langle fh, T\rangle$ we know that $[fh, v] \in T$. Therefore a lower bound for $\kappa(fh)$ is given by

$$\begin{aligned} \kappa(fh) &\geq |[fh, T]| - |[fh, T] \cap C_T(fh)| \\ &\geq |[fh, T]| - |C_T(fh)| \\ &\geq |G|^{r-1} - |G|. \end{aligned}$$

This establishes the proposition. ■

To connect the structure of G to the possible values of κ we will need the two facts stated below. These are extensions of exercises found in Herstein [20]. Part (i) is a special case of the existence of Frobenius kernels (see Subsection 3.3.4 for the definition).

Lemma 2.2.9. *Let G be a finite group.*

1. *If there is a self-centralizing subgroup within G of order 2 then G is odd-dihedral.*
2. *If more than $3/4$ of the elements of G are involutions then G is an elementary abelian 2-group.*

Proof. For part (1), let x be an element of order two in G with $\langle x\rangle$ self-centralizing. Let X denote the conjugacy class of x, then $|X| = |G|/2$. If $H = G \setminus X$ then $|H| = |G|/2$ and $1 \in H$. The goal is to show that H is a subgroup of G.

To this end let $u \in X$ and suppose that $uX \cap X \neq \emptyset$. Then there exists $v \in X$ with $uv \in X$ so that $u^2 = v^2 = (uv)^2 = 1$; hence u and v commute. However u is a conjugate of x so $\langle u\rangle$ is also self-centralizing. Thus $v = u$ and $uv = u^2 = 1$, which contradicts the fact that $uv \in X$. It follows that $uX \cap X = \emptyset$ for all $u \in X$ and hence $uX \subseteq H$.

By size considerations we have that the disjoint union of uX with X is all of G, and hence $uX = H$ and $XX = H$. Then

$$HH = Xu \cdot uX = XX = H$$

and H is indeed a subgroup of index 2 in G. This implies H is normal in G and $G = H \rtimes \langle x\rangle$. Using $H = xX$ we see that every element of H is of the form $h = xy$ for some $y \in X$. Thus, since $x^2 = 1$,

$$h^x = (xy)^x = xxyx = yx = (xy)^{-1} = h^{-1}.$$

It follows that x inverts all elements of H, and therefore H is abelian. Since $\langle x\rangle$ is self-centralizing, we see that H contains no element of order 2. Thus $|H|$ is odd and G is an odd-dihedral group.

To prove part (2) let $E = \{x \in G \mid x^2 = 1\}$. If $u \in E$ then by assumption

$$|E| = |uE| > \frac{3}{4}|G|,$$

so that $|uE \cap E| > |G|/2$. In other words, there exists $V \subseteq E$ with $|V| > |G|/2$ and with $uV \subseteq E$. If $v \in V$ then $u^2 = v^2 = (uv)^2 = 1$; hence u and v commute. Thus $V \subseteq C_G(u)$ with $|V| > |G|/2$. Since $C_G(u)$ is a subgroup of G we have that $C_G(u) = G$. We have therefore shown that E is central in G and clearly E generates G, as $|E| > |G|/2$. We then conclude that G is an elementary abelian 2-group. ■

Now we are ready to connect the possible values of κ the structure of G in a wreath product W.

Proposition 2.2.10. *Let H be a permutation group of finite degree n acting on a set Ω and let G be a finite group. Write $W = G \wr H$ and suppose B is the base of $G \wr H$. Let $\alpha \in \Omega$.*

1. *If f is a non-trivial coordinate element of type α in $G \wr H$ then $\kappa(f) < |G|/2$. More specifically $\kappa(f) < |G : C_G(f(\alpha))|$.*
2. *If there exists f a coordinate element of $G \wr H$ with $|f| = 2$ and $\kappa(f) \geq |G|/4$ then G is odd-dihedral.*
3. *If fh is a non-base element with $\kappa(fh) \leq |G|/2$ then h is an involution.*
4. *If there exists a non-base element fh with $\kappa(fh) < |G|/4$ then G is an elementary abelian 2-group.*

Proof. Suppose f is a coordinate element of type $\alpha \in \Omega$ and that $f(\alpha) = x$ for some non-identity $x \in G$. Since the conjugates of f are all coordinate elements, the only conjugates with which f could possibly not commute are those of the same type. Therefore

$$\{x^g \mid g \in G, [x, x^g] \neq 1\} \subset x^G,$$

and hence

$$\kappa(f) = |\{x^y \mid y \in G, [x, x^y] \neq 1\}| < |G : C_G(x)|.$$

Observe $|C_G(x)| \geq |\langle x \rangle| \geq 2$. Then $\kappa(f) < |G|/2$, completing the proof of part (1).

Suppose further that $|f| = 2$ and $\kappa(f) \geq |G|/4$. Then $|x| = 2$. In this case

$$\frac{|G|}{4} \leq \kappa(f) < \frac{|G|}{|C_G(x)|} \leq \frac{|G|}{2}.$$

Therefore $|C_G(x)| = 2$ and by Proposition 2.2.9 we have that G is odd-dihedral with self-centralizing subgroup $\langle x \rangle$.

For part (3), let fh be a non-base element of W. Then h is non-trivial and has an orbit $\mathcal{O}$ with size $r > 1$. Let $T = \{b \in B \mid b(\omega) = 1 \text{ if } \omega \notin \mathcal{O}\}$ and let t be the number of orbits of h. From Proposition 2.2.8 we know that

$$\kappa(fh) \geq |G|^{r-1} - |G|.$$

Yet we assume $\kappa(fh) \leq |G|/2$. Since $|G| > \frac{3}{2}$, we have

$$\begin{aligned} \tfrac{|G|}{2} \geq |G|^{r-1} - |G| &\implies \tfrac{3}{2}|G| \geq |G|^{r-1} \\ &\implies |G|^2 > |G|^{r-1} \\ &\implies r = 1, 2. \end{aligned}$$

As $r > 1$ we have that $r = 2$. Hence any non-singleton orbit of h is of size 2, meaning that h is an involution. This establishes part (3).

Moving on to part (4), we assume fh is a non-base element of W with $\kappa(fh) < |G|/4$. Part (3) tells us that $h^2 = 1$. Let $\mathcal{O} = \{\alpha, \alpha h\}$ be an orbit of h with size $r = 2$ and let T be as before. Since $r = 2$, Proposition 2.2.8 established

$$\kappa(fh) \geq |G| - |[fh, T] \cap C_T(fh)|.$$

Now our goal is to put a bound on the cardinality of the intersection in the above inequality and use the resulting information to force more than 3/4 of the elements of G to be involutions. Proposition 2.2.9 would then give the desired result.

We begin by considering the elements in $C_T(fh)$. If $b \in C_T(fh)$ and $b(\alpha) = y$ for some $y \in G$ then we also know the possibilities for y are limited to the elements of $C_G(f(\alpha)f(\alpha h))$ by Proposition 2.2.6. To simplify the notation, let $u = f(\alpha)$ and $v = f(\alpha h)$. Thus $|C_T(fh)| \leq |C_G(uv)|$.

We will show that uv must be central in G. Supposing otherwise we can improve on the lower bound from Proposition 2.2.8. Since $C_G(uv) \leq |G|/2$, the number of conjugates fh has under T is

$$|T : C_T(fh)| > \frac{|G|^2}{\frac{|G|}{2}} = 2|G|.$$

At most $|C_T(fh)| \leq |G|/2$ of these can commute with fh. Hence

$$\begin{aligned} \kappa(fh) &\geq |[fh, T]| - |C_T(fh)| \\ &\geq 2|G| - \tfrac{|G|}{2} \\ &= \tfrac{3}{2}|G|. \end{aligned}$$

This contradicts the fact that $\kappa(fh) < \frac{1}{4}|G|$. Therefore uv must be central in G. It follows that we can choose any $y \in G$ to make a function $b \in T$ that commutes with fh.

Next we examine the commutators $[fh, T]$. For $g \in T$ these are given by

$$[fh, g](\omega) = (g^{-1})^{fh} g(\omega) = \begin{cases} 1 & \text{if } \omega \notin \mathcal{O} \\ (g(\alpha h)^v)^{-1} g(\alpha) & \text{if } \omega = \alpha \\ (g(\alpha)^u)^{-1} g(\alpha h) & \text{if } \omega = \alpha h \end{cases} .$$

To make $[fh, g] \in C_T(fh)$ we need a $y \in G$ such that

$$y = (g(\alpha h)^v)^{-1} g(\alpha) \qquad \text{and} \qquad (g(\alpha)^u)^{-1} g(\alpha h) = b(\alpha h) = y^{f(\alpha)} = y^u.$$

Any y that satisfies the above equation for u and v will give an element $b \in C_T(fh) \cap [fh, T]$.

Now that we know what such an element must look like, let us further examine y. Remembering that $uv \in \mathrm{Z}(G)$ we can calculate:

$$\begin{aligned} y &= y^{uv} \\ &= [g(\alpha)^{u^{-1}} g(\alpha h)]^v \\ &= [g(\alpha)^{uv}]^{-1} g(\alpha h)^v \\ &= g(\alpha)^{-1} g(\alpha h)^v \\ &= y^{-1}. \end{aligned}$$

In other words, y is an involution. Therefore all elements of the set $[fh, T] \cap C_T(fh)$ are of the form

$$b(\omega) = \begin{cases} y & \text{if } \omega = \alpha \\ y^{f(\alpha)} & \text{if } \omega = \alpha h \\ 1 & \text{otherwise} \end{cases}$$

where y is an involution in G. Let $\mathcal{E}$ be the number of involutions in G.

We now have

$$\begin{aligned} \kappa(fh) &\geq |[fh, T]| - |[fh, T] \cap C_T(fh)| \\ &\geq |G| - \mathcal{E}. \end{aligned}$$

Yet our assumption is that $\kappa(fh) > \frac{1}{4}|G|$. Now $\frac{1}{4}|G| > |G| - \mathcal{E}$; hence $\mathcal{E} > \frac{3}{4}|G|$. Proposition 2.2.9 yields the desired result, that G is an elementary abelian 2-group. ■

Proposition 2.2.10 clearly relates the value of κ with the structure of G. This information is similar to what Neumann and Bodnarchuk were describing in their separate investigations. However the introduction of the function κ and the clear connection κ has with the structure of G clarifies the calculations and hence the proofs.

One last proposition describing the structure of W is required before we can proceed. Recall that by the definition of an odd-dihedral group, if G is one such group then $\mathbb{O}^2(G)$ is equal to the odd-order abelian subgroup of index 2.

Proposition 2.2.11. *The smallest subgroup of B that has a 2-group quotient, $\mathbb{O}^2(B)$, is equal to $\prod \mathbb{O}^2(G)$ and is characteristic in W.*

Proof. That $\mathbb{O}^2(B) = \prod \mathbb{O}^2(G)$ is clear. Instead we prove that $\mathbb{O}^2(B)$ is characteristic in W. Let $S = \langle g^2 \mid g \in G\rangle$. Then $S \trianglelefteq G$ with an elementary abelian 2-group as quotient. Define

$$K = \langle w^2 \mid \kappa(w) \leq \frac{1}{2}|G|\rangle,$$

so that K is a characteristic subgroup of W. Part (2) of Proposition 2.2.10 tells us that if $w \in W - B$ and $\kappa(w) \leq \frac{1}{2}|G|$ then w^2 is an element of B. We see that $K \leq B$.

Moreover, since $S \times S \times S \times \cdots \times S$ is the group generated by the set of the squares of all coordinate elements, we know K contains $S \times S \times S \times \cdots \times S$ by Proposition 2.2.10 (1). Thus

$$\prod S \leq K \leq B = \prod G$$

and, since G/S is a 2-group, the quotient B/K is also a 2-group. It follows that $\mathbb{O}^2(B) = \mathbb{O}^2(K)$ is characteristic in W. ■

With this information established, we are now ready to investigate the automorphisms of $G \wr H$ in the case that the base is not characteristic. In particular, if an automorphism of W moves the base B then G must have one of two possible structures.

Theorem 2.2.12. *Let G be a finite group and let H be a permutation group of degree $n < \infty$. Let $W = G \wr H$ be the wreath product of G with H and suppose B is the base of W. If B is not characteristic in W then G is either an elementary abelian 2-group or an odd-dihedral group.*

Proof. Suppose that B is not characteristic in W and let φ be an automorphism of W for which $B^\varphi \neq B$. We define $L \leq W$ by

$$L = \langle w \in W \mid \kappa(w) < \frac{1}{4}|G|\rangle.$$

Since L is a characteristic subgroup of W, we know that $L \neq B$. Then there are two cases: $L \nleq B$ or $B \nleq L$.

First suppose that $L \nleq B$. Then there exists $w \in L - B$, meaning that $\kappa(w) < \frac{1}{4}|G|$. By Proposition 2.2.10 (4), the group G is an elementary abelian 2-group.

Now suppose that $B \nleq L$. We then know that there exists a non-identity coordinate element $f \notin L$. Hence $\kappa(f) \geq \frac{1}{4}|G|$. From Proposition 2.2.10(1) we have an upper bound on $\kappa(f)$ as well. Let f be of type $\alpha \in \Omega$ and suppose $f(\alpha) = x$, where x is a non-trivial element of G. This gives:

$$\frac{|G|}{4} \leq \kappa(f) < \frac{|G|}{|C_G(x)|} \leq \frac{|G|}{2}.$$

Therefore $C_G(x)$ has order 2 or 3.

If $|C_G(x)| = 2$ then x is a self-centralizing element of G with order 2. Proposition 2.2.9 then gives that G is odd-dihedral. We will show that if $C_G(x)$ has order 3 then G is odd-dihedral, as well.

Suppose that $|C_G(x)| = 3$ and $N_G(\langle x \rangle) = \langle x \rangle$. Then x and x^{-1} are not conjugate to one another. However, each one has $\frac{1}{3}|G|$ conjugates and these conjugates all have odd order. Then

$$|\mathbb{O}^2(G)| \geq \frac{2}{3}|G|.$$

This implies that $\mathbb{O}^2(G) = G$. Hence Proposition 2.2.11 gives that $B = \mathbb{O}^2(B)$ is a characteristic subgroup of W. Yet we know B is not characteristic, so we must assume that $N_G(\langle x \rangle) > \langle x \rangle$.

In this case $N_G(\langle x \rangle) \cong S_3$, so there must exist elements y_1, y_2 of order 2 such that $x = y_1 y_2$. Define b_1, b_2 coordinate elements of type α (the same type as f) in W by

$$b_m(\omega) = \begin{cases} 1 & \text{if } \omega \neq \alpha \\ y_m & \text{if } \omega = \alpha \end{cases}$$

where $m = 1, 2$. Now $|b_m| = 2$ for each value of m, so either $b_m \in L$ or G is odd-dihedral by Proposition 2.2.10(2). If we assume that $b_1, b_2 \in L$, then since $b_1 b_2 = f$ we have then that $f \in L$. This contradiction tells us that if $C_G(x)$ has order three then G must also be odd-dihedral.

We have therefore shown that if B is not characteristic in W then G is either an elementary abelian 2-group or an odd-dihedral group. ■

The definition of odd-dihedral does allow the odd-order abelian subgroup to be trivial, and hence C_2 is both an elementary abelian 2-group and an odd-dihedral group. We next show that if G is elementary abelian and the base of $G \wr H$ is not characteristic then $G = C_2$. To do this we switch tactics use permutation group theory to consider the action of B^{φ}, the image of B under an automorphism φ, as a permutation group acting on the same set as H.

Proposition 2.2.13. *Let G and H be finite groups with H a permutation group of degree n on the set Ω. If the base B is not characteristic in $G \wr H$ and G is an elementary abelian 2-group then $G = C_2$.*

Proof. The order of B is $|G|^n$ where $|G|$ is now a power of 2. Therefore B is a self-centralizing elementary abelian 2-subgroup of W. Since B is not characteristic in W there exists an automorphism, φ, of W with $M = B^{\varphi} \neq B$.

Then M is also a self-centralizing elementary abelian normal 2-subgroup in W. Notice that $\overline{M} = M/(M \cap B)$ is permutation isomorphic to MB/B, which is a non-identity normal abelian subgroup of $W/B \cong H$. Suppose that the group $\overline{M}$ has t orbits, say $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_t$ with sizes $m_1, m_2, \ldots, m_t$ respectively in its action on Ω. Then

$$m_1 + m_2 + \cdots + m_t = n.$$

Furthermore some $m_i > 1$ since H, and hence $\overline{M}$, is faithful and $\overline{M}$ is non-trivial.

We will first obtain an upper bound on $|M|$ and then use that upper bound to determine that $|G| = 2$. We start with $|M \cap B|$. The subgroup M is self-centralizing so $C_W(M) = M$ and hence

$$C_B(M) = C_W(M) \cap B = M \cap B.$$

Furthermore M permutes the n factors of B in t orbits. From Proposition 2.2.7 we have

$$|M \cap B| = |C_B(M)| \leq |G|^t.$$

Moving on to consider $|M : M \cap B|$ we let $M_i = M_{(\mathcal{O}_i)}$, the subgroup of M whose elements fix each point in $\mathcal{O}_i$. Then M/M_i is a transitive abelian group of permutations on $\{\mathcal{O}_i\}$, implying that M/M_i is regular in this action. Hence

$$|M : M_i| = |M/M_i| = |\mathcal{O}_i| = m_i$$

for all i. It follows by induction that

$$|M : \bigcap_{i=1}^{t} M_i| \le \prod_{i=1}^{t} m_i.$$

Yet $\bigcap_{i=1}^{t} M_i$ is the subgroup of M that fixes all n points, so

$$\bigcap_{i=1}^{t} M_i = M \cap B$$

and we conclude that

$$|M : M \cap B| \le \prod_{i=1}^{t} m_i.$$

It follows that

$$\begin{aligned} |M| &= |M \cap B||M : M \cap B| \\ &\le |G|^t \textstyle\prod_{i=1}^{t} m_i \\ &= \textstyle\prod_{i=1}^{t} |G| m_i. \end{aligned}$$

Hence

$$\begin{aligned} \textstyle\prod_{i=1}^{t} |G| m_i &\ge |M| \\ &= |G|^n \\ &= \textstyle\prod_{i=1}^{t} |G|^{m_i}. \end{aligned}$$

Now to develop our understanding of $|G|$, set

$$a_i = \frac{|G|m_i}{|G|^{m_i}}.$$

It's clear then that $\prod_{i=1}^{t} a_i \ge 1$. Notice that if $m_i = 1$ then $a_i = 1$. There does exist, however, an m_i that is not equal to one. If there exists an $a_k < 1$ then there must exist $a_j > 1$, in order for the product of all a_i to be at least 1. Hence we may assume that there exists j such that $m_j > 1$ and $a_j \ge 1$.

Then $m_j \ge |G|^{m_j - 1} \ge 2^{m_j - 1}$ and so $2m_j \ge 2^{m_j}$. Hence we see first that $m_j = 2$ and then that

$$a_j = \frac{2|G|}{|G|^2} = \frac{2}{|G|} \ge 1.$$

Thus $a_j = 1$ and $|G| = 2$ as required. ■

We can further develop the proof of Proposition 2.2.13 to describe the structure of H, providing new insight into the image of B when this subgroup is not characteristic. We will maintain the notation regarding the point-wise stabilizer of a set. Namely, if K is a permutation group then $K_{(\mathcal{O})}$ is the subgroup of permutations of K that fix each point in $\mathcal{O}$.

Corollary 2.2.14. *Let H be a finite permutation group of degree n acting on Ω. Suppose that the base B of $C_2 \wr H$ is not characteristic and let $\varphi \in Aut\ (C_2 \wr H)$ such that $M = B^{\varphi} \neq B$. Let $\overline{M} = M/M \cap B$. Then:*

1. *The subgroup M acts on the set Ω with orbits $\mathcal{O}_i$ of size 1 or 2.*
2. *The quotient $\overline{M}$ is isomorphic to the direct product with factors $M/M_{(\mathcal{O}_i)}$ for distinct non-trivial orbits $\mathcal{O}_i$ of M.*
3. *Alternatively, if we consider only non-trivial orbits $\mathcal{O}_i$ of M then $\overline{M}$ is isomorphic to the direct product $\prod M_{(\mathcal{Q}_i)}$ for $Q_i = \Omega - \mathcal{O}_i$. Consequently $M/M \cap B$ is generated by commuting transpositions.*
4. *Lastly, if there exists an orbit $\mathcal{O}$ of H with an element $\alpha \in \mathcal{O}$ that is not fixed by $\overline{M}$ then $\overline{M}$ can fix no element of $\mathcal{O}$.*

Proof. We utilize the same notation as in the proof of Proposition 2.2.13, which proved that the orbits of $\overline{M} = M/M \cap B$ were of size 1 or 2 with at least one orbit of size 2. Moreover, the proof showed that each $a_i = 1$ so that

$$|M| = \prod_{i=1}^{t} |G|m_i = \prod_{i=1}^{t} |G|^{m_i}$$

and $|M \cap B| = |G|^t$. Hence

$$|\overline{M}| = |M : M \cap B| = \prod_{i=1}^{t} m_i,$$

where m_i is the size of the orbit $\mathcal{O}_i$. Recall that $M_i = M_{(\mathcal{O}_i)}$ is the subgroup of M that fixes all points in $\mathcal{O}_i$ point-wise, and hence M/M_i is a regular abelian group.

Define a homomorphism $\psi : M \to \prod M/M_i$, where for $m \in M$ the i^{th} component of $\psi(m)$ is the restriction of m to $\mathcal{O}_i$. In order for $\psi(m)$ to have a i^{th} component, it must be the case that m acts trivially on $\mathcal{O}_i$. Hence $m \in M_i$. Thus the kernel of this map is $\bigcap M_i = M \cap B$. Therefore, due to order considerations, $\overline{M} \cong \prod M/M_i$ and this establishes part (2) of the corollary.

Now let $\mathcal{Q}_i = \Omega - \mathcal{O}_i$. Then $M_{(\mathcal{Q}_1)}$ is the subgroup of all elements in M which fix each point in $\mathcal{Q}_i$, and hence

$$M_{(\mathcal{Q}_i)} = \bigcap_{j \neq i} M_{(\mathcal{O}_j)} = \bigcap_{j \neq i} M_i.$$

Thus the non-identity elements of $M_{(\mathcal{Q}_i)}$ are transpositions on $\mathcal{O}_i$ that fix all other points. Then if x_i is an element of $\mathcal{Q}_i$ and x_j is an element of $\mathcal{Q}_j$ for $i \neq j$, we have that x_i and x_j have disjoint support and therefore commute. Then $M_{(\mathcal{Q}_i)}M_{(\mathcal{Q}_j)} = M_{(\mathcal{Q}_j)}M_{(\mathcal{Q}_i)}$, implying that the product is a subgroup of M. Moreover, this subgroup is a direct product. Since this is true for all i, j with $1 \leq i, j \leq t$ where $i \neq j$, we actually have that

$$\prod_{i=1}^{t} M_{(\mathcal{Q}_i)} \leq M.$$

Define a homomorphism $\sigma : M \to \prod M_{(\mathcal{Q}_i)}$, where for $m \in M$ the i^{th} component of $\sigma(m)$ is the restriction of m to $\mathcal{Q}_i$. As each element of $\prod M_{(\mathcal{Q}_i)}$ is in the domain, the homomorphism is surjective. Moreover $\mathcal{O}_i$ are orbits so they intersect trivially. Thus the kernel of σ is the subgroup of all elements in M which fix each point in Ω, meaning ker $\sigma = M \cap B$. This establishes part (3) of the corollary.

Remember that the actions of $\overline{M} = M/M \cap B$ and MB/B are permutation equivalent and these two groups are isomorphic with the latter a normal subgroup of $W/B \cong H$. Therefore $\overline{M}$ acts equivalently to a normal subgroup of H. If H is faithful on an orbit $\mathcal{O}$ then so is $\overline{M}$. Moreover as a normal subgroup, we know that if there exists $\alpha \in \mathcal{O}$ that is fixed by $\overline{M}$ then every point in $\mathcal{O}$ is fixed by $\overline{M}$. Since $\overline{M}$ is faithful, it cannot be in the kernel of the action and therefore there cannot exist fixed points in $\mathcal{O}$. ■

Of course, the main result of the section follows from Proposition 2.2.13.

Theorem 2.2.15. *Let G be a finite group and H a finite permutation group. If the base of $G \wr H$ is not characteristic then G is odd-dihedral.*

Yet the developments of Corollary 2.2.13 give rise to the following additional proposition.

Proposition 2.2.16. *Let G be a finite group and H a finite permutation group of degree n. Let $\varphi \in$ Aut $(G \wr H)$ be such that $B^\varphi \neq B$. Then $BB^\varphi = B \rtimes H_\varphi$ where H_φ is a non-trivial normal subgroup of H that is an elementary abelian 2-group generated by commuting transpositions. The orbits of H_φ are of size 1 or 2, with at least one non-trivial orbit. Moreover, if in H_φ there is a point α in an orbit $\mathcal{O}$ of H that H_φ does not fix then H_φ cannot fix any point in $\mathcal{O}$.*

Proof. From Theorem 2.2.15 we know that G is an odd-dihedral group. Suppose first that $G \cong C_2$ and let $M = B^{\varphi}$. The group $M/M \cap B$ is permutation isomorphic to a non-trivial normal subgroup of $W/B \cong H$, say H_{φ}, and since M is elementary abelian we have that M splits over $M \cap B$. Then

$$1 \neq H_{\varphi} \cong M/(M \cap B) \cong BB^{\varphi}/B$$

and thus $BB^{\varphi} \cong B \rtimes H_{\varphi} = G \wr H_{\varphi}$. Furthermore Corollary 2.2.14 yields that $M/M \cap B$, and hence H_{φ}, is an elementary abelian 2-group generated by commuting transpositions. Lastly, if H is faithful on an orbit then $M/M \cap B$, and hence H_{φ}, cannot fix any points in that orbit. Thus we have established the proposition in the case that $G = C_2$.

Suppose now that G is odd-dihedral with a non-trivial abelian normal subgroup of index 2. This subgroup is the smallest normal subgroup of G with a 2-group quotient, thus $G/\mathbb{O}^2(G) \cong C_2$. By Proposition 2.2.11 the subgroup $\mathbb{O}^2(B) = \prod \mathbb{O}^2(G)$ is characteristic in W and has quotient

$$W/\mathbb{O}^2(B) \cong (G/\mathbb{O}^2(G)) \wr H$$

with base $B/\mathbb{O}^2(B)$. Then B^{φ} contains $\mathbb{O}^2(B)$ so

$$(BB^{\varphi})/\mathbb{O}^2(B) = (B/\mathbb{O}^2(B))(B^{\varphi}/\mathbb{O}^2(B)) \cong (B/\mathbb{O}^2(B))(B/\mathbb{O}^2(B))^{\varphi}.$$

In fact, φ can be thought of as an automorphism of $W/\mathbb{O}^2(B)$ and as such will not stabilize the base $B/\mathbb{O}^2(B)$. Since $G/\mathbb{O}^2(G)$ has order 2, now $BB^{\varphi}/\mathbb{O}^2(B)$ as an elementary abelian group will split into

$$(BB^{\varphi})/\mathbb{O}^2(B) = (B/\mathbb{O}^2(B)) \rtimes H_{\varphi}$$

where H_{φ} is a non-identity normal subgroup of H generated by transpositions. Thus $BB^{\varphi} = B \rtimes H_{\varphi} = G \wr H_{\varphi}$ and the result follows. ∎

Corollary 2.2.17. *If the base B is not characteristic in $G \wr H$ then there exists a characteristic subgroup N of $G \wr H$ such that $B < N$ and N/B is generated by normal elementary abelian 2-subgroups of G/B.*

Proof. Let $W = G \wr H$. In view of Proposition 2.2.11, the subgroup $\mathbb{O}^2(B) = \prod \mathbb{O}^2(G)$ is characteristic in W and is properly contained in B. Since $W/\mathbb{O}^2(B) = (G/\mathbb{O}^2(G)) \wr H$

with base group $G/\mathbb{O}^2(G)$, it suffices to assume that $\mathbb{O}^2(B) = 1$. Then $G = C_2$ and B is an elementary abelian 2-group.

Let N be generated by all of the elementary abelian normal 2-subgroups of W. Then N will be characteristic in W, it will contain B, and the quotient N/B will be generated by elementary abelian normal 2-subgroups of W/N. Since B is not characteristic in G, we must have that $B < N$. ■

We note that N/B may not be abelian.

2.2.2 Specific Actions

More can be said about the structure of $W = G \wr H$ when we have additional information about the action of H on Ω. One obvious consequence is the following, when $H \leq A_n$.

Corollary 2.2.18. *Let $W = G \wr H$ be the wreath product of the group G by $H \leq S_n$. If $H \leq A_n$ then B is characteristic in W.*

Proof. The group A_n does not contain transpositions, and hence H cannot be as described in Proposition 2.2.16. We must then conclude that B is characteristic. ■

We additionally can extend Theorem 2.2.2. Recall that the action H is semiregular if the point stabilizers are all trivial.

Corollary 2.2.19. *Let $W = G \wr H$ where G is odd-dihedral and H is a faithful permutation group of degree n acting on a set Ω. If H is either semiregular or primitive, and if the base group B is not characteristic in W, then $n = 2$ and $H \cong C_2 = S_2$.*

Proof. Let φ be an automorphism of W that does not stabilize B. By Proposition 2.2.16, there is a subgroup H_φ of H that is a non-identity normal abelian subgroup generated by commuting transpositions and the orbits of H_φ are all of size 1 or 2. If H is semiregular, then H_φ is semiregular as well and therefore the orbits of H_φ must all be non-trivial. The Orbit-Stabilizer Theorem [9, Proposition 4.1.2] then gives that $H_\varphi \cong C_2$. By our definition of transposition, H_φ then moves exactly two points in Ω and fixes all the other points. Yet H is semiregular and no non-trivial element of H fixes a point in Ω. Hence $n = 2$ and $H = H_\varphi \cong C_2$.

If H is primitive then the transitive abelian normal subgroup H_φ would be regular. Yet H_φ is generated by transpositions – that cannot have fixed points due to the trivial point stabilizers – and so $n = 2$. Hence $H \cong C_2 = S_2$. ■

In particular, if H is transitive on Ω then there exists only 1 orbit. Hence if the base of $G \wr H$ is not characteristic then the subgroup H_φ from Proposition 2.2.16 must move every single point in Ω. Hence H has an elementary abelian normal 2-subgroup that moves all points and is generated by commuting transpositions. This led us to explore such subgroups of S_n, which gave the following result.

Lemma 2.2.20. *Let P be an elementary abelian 2-subgroup of S_n that moves all points and is generated by commuting transpositions. Then $n = 2m$ is even, $|P| = 2^m$, and*

$$N_{S_n}(P) = P \rtimes K = C_2 \wr K$$

where $K \cong S_m$ transitively permutes the generators of P by conjugation. If x is a transposition in P then

$$P = \langle x^\varphi \mid \varphi \in N_{S_n}(P) \rangle.$$

Proof. Since P is a 2-group fixing no point, it follows that all of its orbits are of even size. Hence $n = 2m$ is even. Without loss of generality we can assume that P is generated by the m commuting transpositions $(1\ m+1), (2\ m+2), \ldots, (m\ m+m)$. Hence $|P| = 2^m$. Let us write $\Delta_i = \{i, m+i\}$ for $i = 1, 2, \ldots, m$ so that these sets form a partition of $\Omega = \{1, 2, \ldots, n\}$.

Now it's easy to see that the transpositions that generate P are the only transpositions in P, since in our definition transpositions move exactly two points in Ω. Thus $Q = N_{S_n}(P)$ permutes these generators by conjugation. Note also that, for $g \in S_n$, we have $(i\ m+i)^g = (j\ m+j)$ if and only if $\Delta_i g = \Delta_j$. In particular Q is precisely the set of all elements of S_n that permute the blocks $\Delta_1, \Delta_2, \ldots, \Delta_m$. Let $\theta : Q \to S_m$ be the homomorphism describing this permutation action.

Let $K \cong S_m$ be the subgroup of S_n consisting of all permutations that act the same on $\{1, 2, \ldots, m\}$ as it does on $\{m+1, m+2, \ldots, m+m\}$. Specifically, if $k \in S_n$ acts on $\{1, 2, \ldots, m\}$ then $k \in K$ if and only if $ik = j$ implies $(m+i)k = m+j$. Now it is clear that K faithfully permutes the blocks Δ_i and that it acts as the full symmetric group on the set of blocks. In particular, we know that $K \leq Q$ and that $\theta(K) = S_m$. Thus

$\theta(Q) = \theta(K) = S_m$. Furthermore, the kernel of θ is generated by all transpositions of the form $(i\ m+i)$ and hence the kernel of θ is P. As $P \cap K = 1$, by order considerations if follows that $TK = Q$ so $Q = P \rtimes K$ where K transitively permutes the m generators of P by conjugation. With this we see that Q is naturally isomorphic to $C_2 \wr K$ where C_2 is any subgroup of P generated by a single transposition. ∎

Before we prove the main result of this section, we provide a independent proof that if G is odd-dihedral then the base of $G \wr C_2$ is not characteristic. While this fact can be proved using Theorem 2.2.2, the proof we give here is interesting because it relies only on facts about odd-dihedral groups and D_8.

Proposition 2.2.21. *If G is an odd-dihedral group then $G \wr C_2$ does not have a characteristic base.*

Proof. Let $W = G \wr C_2$ and B be the base of W. Then $W = B \rtimes \langle t \rangle$. We can also write $B = G_1 \times G_2$, where $G_i \cong G$ for $i = 1, 2$ and $G_1^t = G_2$. Since G is odd-dihedral there exists an element $x \in G_1$ such that $G_1 = \mathbb{O}^2(G_1) \rtimes \langle x \rangle$ where $\mathbb{O}^2(G_1)$ is an odd-order abelian subgroup upon which x acts by inversion. Similarly there exists $y \in G_2$ such that $G_2 = \mathbb{O}^2(G_2) \rtimes \langle y \rangle$ where $\mathbb{O}^2(G_2)$ and y have the same properties as $\mathbb{O}^2(G_1)$ and x. Observe that $\mathbb{O}^2(G_2) = (\mathbb{O}^2(G_1))^t$ and $y = x^t$.

Using the fact that G_1 and G_2 commute it follows that $W = V \rtimes D$ where $V = \mathbb{O}^2(G_1) \times \mathbb{O}^2(G_2)$ is an odd-order abelian group and $D = \langle x, y, t \rangle$. It also follows that x and y commute, hence the subgroup $\langle x, y \rangle$ is a normal elementary abelian subgroup of order 4 in D. Thus $D = \langle x, y \rangle \rtimes \langle t \rangle$ with $x^t = y$. Hence D is a dihedral group of order 8 and $D = \langle x, t \mid x^2 = t^2 = (xt)^4 = 1 \rangle$ as in Section 1.2. Furthermore $B = V \rtimes \langle x, y \rangle$.

Since x inverts $\mathbb{O}^2(G_1)$ and centralizes $\mathbb{O}^2(G_2)$, and oppositely for y, it follows that xy inverts all elements of B. Thus W satisfies the hypotheses of Proposition 1.2.4. In particular if φ is the automorphism of D that interchanges x and t then by Proposition 1.2.4 the automorphism φ extends to an automorphism φ^* of W. But $x \in B$ and $\varphi^*(x) = t \notin B$. Thus the extension of φ does not stabilize B and hence B is not characteristic in W. ∎

We will now use this fact to completely determine when the base of $G \wr H$ is not characteristic, for a transitive group H.

Theorem 2.2.22. *Let $W = G \wr H$ be the wreath product of a group G by H, a transitive permutation group of degree n acting on Ω. Then the base group B is not characteristic in W if and only if G is an odd-dihedral group and $H = C_2 \wr K$ where $n = 2m$ is even and K is a permutation group of degree m. Furthermore the subgroup $\prod(G \wr C_2)$, with m factors, is characteristic in W. (This is the "N" in Corollary 2.2.17.)*

Proof. Suppose first that B is not characteristic in W; then G is odd-dihedral. Since H is a transitive permutation group of degree n, it is isomorphic to a subgroup H_1 of S_n. By Proposition 2.2.16 H has a normal elementary abelian 2-subgroup P generated by commuting transpositions and moving all points of Ω. Then P is isomorphic to a subgroup P_1 of S_n with the same properties, including $P_1 \trianglelefteq H_1$.

Thus by Lemma 2.2.20 there is a positive integer m such that $n = 2m$ and $N_{S_n}(P_1) = P_1 \rtimes L_1$ where $L_1 \cong S_m$. Since $P_1 \trianglelefteq H_1$ we have that $H_1 \leq N_{S_n}(P_1) = P_1 \rtimes L_1$. Therefore $H_1 = P_1 \rtimes K_1$ where $K_1 = L_1 \cap H_1$ is a permutation group of degree m.

Moving back into H we have that $H = P \rtimes K$ where $K \cong K_1$ is a permutation group of degree m. Then

$$W = B \rtimes H = B \rtimes (P \rtimes K) = (B \rtimes P) \rtimes K.$$

We note that $B \rtimes P = \prod(G \wr C_2)$ where the various C_2 groups correspond to the $n/2 = m$ transpositions in P. With this we see that

$$W = (B \rtimes P) \rtimes K = (G \wr C_2) \wr K = G \wr (C_2 \wr K).$$

Conversely, assume that G is an odd-dihedral group and suppose $W = G \wr (C_2 \wr K)$ for K, a permutation group of degree $m = n/2$. Then using the associativity of the permutational wreath product, $W = (G \wr C_2) \wr K$. Let

$$B_1 = \prod_{i=1}^{m} (G \wr C_2)$$

so that $W = B_1 \rtimes K$. By Proposition 2.2.21 we know that $G \wr C_2$ admits an automorphism φ that moves its base $G \times G$. Let us extend φ to an automorphism of W by defining φ^* to act trivially on K and restrict to φ on each factor of $G \wr C_2$. In other words, if $(x_1, x_2, \ldots, x_m) \in B_1$ for $x_i \in G \wr C_2$ and $k \in K$ then

$$((x_1, x_2, \ldots, x_m)k)^{\varphi^*} = (x_1^{\varphi^*}, x_2^{\varphi^*}, \ldots, x_m^{\varphi^*})k.$$

If

$$B_2 = \prod_{i=1}^{n} G$$

then we know $B_2^{\varphi^*} \neq B_2$, since $(G \times G)^{\varphi} \neq G \times G$. Therefore the extended automorphism φ^* moves the base group B_2 of W just as φ moves the base $G \times G$ of $G \wr C_2$.

Lastly, $G \wr C_2 = (G \times G) \rtimes C_2$ is not odd-dihedral since G is not of odd order and C_2 does not invert the elements of G. Therefore B_1 is characteristic in $W = (G \wr C_2) \wr K$. ∎

2.3 Automorphism Groups

The automorphism group in the case where the base is characteristic has been extensively studied in [17] and [2]. A discussion of the structure of the automorphism group when the base is characteristic is also presented in [24].

However, there is less understanding of the automorphism group of $G \wr H$ when the base is not characteristic. If the base is not characteristic but H is transitive then G is odd-dihedral and $H = C_2 \wr K$, for some group K. Moreover $W = (G \wr C_2) \wr K$, where the new base is characteristic. Thus when studying the automorphism group of $G \wr H$ for a transitive H we need only examine the automorphism group of $G \wr C_2$ where G is odd-dihedral. If $G = C_2$ then $W \cong D_8$, so we further focus on the case where $G \neq C_2$.

Proposition 2.3.1. *Let G be an odd-dihedral group with abelian odd-order normal subgroup A of index 2 in G. Let $W = G \wr C_2$ and $V = A \times A$. Fix D, a Sylow 2-subgroup of W. The following hold.*

1. *The group $N_V(D) \cap V = 1$, so that V acts faithfully and transitively on the set of Sylow 2-subgroups of W.*
2. *The homomorphism $\varphi : Aut\ W \to Aut\ W/V$ is surjective.*
3. *If V_1 is the isomorphic image of V in $Aut\ W$ under the natural embedding, then $ker\ \varphi = V_1 \rtimes N_{ker\ \varphi}(D)$.*
4. *The complement $N_{ker\ \varphi}(D)$ is isomorphic to $Aut\ A$, acting diagonally on V.*
5. *The order of $Aut\ W$ is equal to $8|Aut\ A||A|^2$.*

Proof. Let $C_2 = \langle t \rangle$, so that $W = G \wr \langle t \rangle$. Let $G = A \rtimes \langle \overline{x} \rangle$ where $\langle \overline{x} \rangle \cong C_2$ and $\overline{x}$ acts on A by inversion. By Proposition 1.2.4 we know that $W = V \rtimes D$ where $D = \langle x, t \rangle \cong D_8$ is a Sylow 2-subgroup of W and $x = (\overline{x}, 1)$.

To begin proving part (1), let us consider the conjugate of $d \in D$ by $v \in V$.

$$\begin{aligned} d^v \in D &\Leftrightarrow d^{-1}d^v \in D \\ &\Leftrightarrow [d, v] \in D \cap V = 1 \end{aligned}$$

So $N_V(D) = C_V(D)$.

To prove that $N_V(D) = 1$, notice that $C_V(x) = 1 \times A$ since x inverts all elements of the first coordinate. Additionally $C_V(t) = \{(a, a) \mid a \in A\}$. Putting these two together gives us the following.

$$C_V(D) \leq C_V(x) \cap C_V(t) = 1$$

The above tells us that $N_V(D) = 1$, meaning that V acts faithfully on the set of Sylow 2-subgroups via conjugation. From Sylow's theorem [9, Theorem 4.1] we know that V acts transitively on the set of Sylow 2-subgroups, which establishes part (1).

As V is a characteristic subgroup we can define $\varphi : \text{Aut } W \to \text{Aut } W/V$ where an automorphism of W now acts on the cosets dV in W/V. Proposition 1.2.4 also tells us that this homomorphism is surjective, yielding part (2).

Let K denote the kernel of φ and let V_1 be the subgroup of inner automorphisms induced by elements of V. Since V acts faithfully on the set of Sylow 2-subgroups, no non-trivial element of V induces the trivial automorphism. Therefore $V \cong V_1$. Of course, V acts trivially on W/V so $V_1 \leq K$. The automorphisms of W normalize V, a characteristic subgroup, and thus they also normalize the group of inner automorphisms induced by elements of V. Thus $V_1 \trianglelefteq \text{Aut } W$, and $V_1 \trianglelefteq K$.

Since V is faithful and transitive in its action on the Sylow 2-subgroups of W, its isomorphic image V_1 is also faithful and transitive. As a transitive normal subgroup of K, we now have

$$K = V_1 \cdot N_K(D).$$

Yet V_1 is faithful so $V_1 \cap N_K(D) = 1$. Therefore K is a semidirect product.

We now consider $N_K(D)$. Suppose that $\psi \in N_K(D)$. Since $\psi \in K$ we know that

$$(dV)^{\psi} = d^{\psi}V = dV$$

for all $d \in D$. This yields that $d^{\psi} = dv$ for some $v \in V$. Yet $\psi \in N_K(D)$, so $d^{\psi} \in D$ implying $v = 1$. Therefore $N_K(D) = C_K(D)$ and ψ will commute with τ_d, conjugation by d for all $d \in D$. Then Equation 1.1 from Chapter 1.2 holds for ψ when φ is the identity automorphism of D, meaning that for all $(v_1, v_2) \in V$ the next equation holds.

$$\psi((v_1, v_2)^d) = (\psi(v_1, v_2))^d \tag{2.2}$$

In particular, Equation 2.2 will hold when $d = x$. Examining Equation 2.2 in this case will lead one to infer that $N_K(D)$ can be embedded in Aut A acting diagonally on V. Conversely, it is easy to check that every such diagonal automorphism satisfies Equation 2.2, so we conclude that $N_K(D) \cong$ Aut A acting diagonally on V.

While φ is surjective with kernel K, we do not know that Aut W splits over K. Thus we can only establish the order of Aut W, which follows.

$$|\text{Aut } W| = |\text{Aut } W/V||V||N_K(D)|$$

The order of Aut W/V is equal to the order of Aut D_8, which is again 8. Then $V = A \times A$, so we may substitute $|A|^2$ for $|V|$ in the formula. As established in part (4), the normalizer in K of D is isomorphic to Aut A. This yields the final result of the proposition. ■

Chapter 3

Complete Groups

Mathematicians have been studying complete groups at least since the nineteenth century. In his 1911 text W. Burnside described a complete group as one that "contains no self-conjugate operation except identity and admits no outer automorphism" [4, Chapter VI, Section 70]. In modern language, we would say that a group is complete if and only if it has a trivial center and the quotient Out $G = \text{Aut } G/\text{Inn } G$ is trivial. It's more common, however, to use the following equivalent definition.

Definition 3.0.2. A group G is complete if $\mathrm{Z}(G) = 1$ and Aut $G =$ Inn G.

We remind readers that the homomorphism sending a group element g to the automorphism defined by conjugation by g maps G onto Inn G. When the center of G is trivial, this map is an isomorphism. Therefore if G is complete then the map is surjective with codomain Aut G, imply that $G \cong \text{Aut } G$. However, this condition alone is not sufficient to produce a complete group. Notice that the automorphism group of D_8 is isomorphic to D_8, yet the dihedral group of order 8 does not have a trivial center.

In the first section of this chapter we will discuss examples, motivation for the study of complete groups, and well-known results. The next section will describe our initial results in the study of complete solvable odd-order groups. The final section will use cohomological techniques to characterize complete wreath products and Frobenius groups, as well as give insight into complete standard wreath products and a few specific examples.

3.1 Background

One of the early results regarding complete groups was proved by Hölder in 1895 [21]. The converse was proved half a century later in 1946 by Baer [1].

Proposition 3.1.1. *A group N is complete if and only for every group G with $N \trianglelefteq G$ there exists a subgroup H of G such that $G = N \times H$.*

In 1911 W. Burnside published his textbook on finite groups, including two sections discussing complete groups. Burnside also credits Hölder as inspiring the result that if the inner automorphism group of a simple group S is characteristic in Aut S then Aut S is complete. Later work showed that Inn S is always characteristic in Aut S for a non-abelian simple group S. Therefore we refine Burnside's result, which provides a plethora of examples of complete groups.

Proposition 3.1.2 (Chapter VI, Section 71 [4])**.** *If S is a non-abelian simple group then Aut S is complete.*

Corollary 3.1.3. *For $n = 5$ or $n \geq 7$, the symmetric group on n points is complete.*

The corollary follows from the proposition since Aut $A_n \cong S_n$ for the pertinent values of n (see [27, Theorem 5.7]). In fact, the automorphism group of A_6 is also complete but it is not isomorphic to S_6. It turns out that the index of S_6 in its automorphism group is 2, so S_6 is not complete. Clearly $S_2 \cong C_2$ cannot be complete, since it is abelian.

On the other hand, it is easy to prove that S_3 is complete, by counting the number of generating sets containing a 2-cycle and a 3-cycle. There are 6 such generating sets. Since an automorphism must send one such set to another, we see that there are at most 6 elements in Aut S_3. Yet of course $\mathrm{Z}(S_3) = 1$, implying that Inn $S_3 \cong S_3$ and therefore there are at least 6 elements in Aut S_3. A more complex but similar argument will yield that S_4 is complete.

Proposition 3.1.4. *The groups S_3 and S_4 are complete.*

Therefore nearly all of the finite symmetric groups are complete groups. Burnside also provided another infinite family of complete groups.

Proposition 3.1.5 (Chapter VI, Section 71 [4])**.** *The holomorph of a finite abelian group of odd-order is complete.*

In particular this proposition implies that if p is an odd prime then $C_p \rtimes C_{p-1}$ is complete. This of course is another way to prove that S_3, the holomorph of C_3, is complete. Note, however, that in a 1953 article W. H. Mills proved that the holomorph of a non-abelian group is never complete [25, p. 428-429]. We will re-visit this information when we discuss complete Frobenius groups later in the chapter. However, Proposition 3.1.5 begs the question: When is the holomorph of an even-order abelian group complete? The answer to this question may be known as the study of holomorphs is quite developed (see [28], [25] for details).

Meanwhile we can construct complete groups from known examples using the following theorem.

Theorem 3.1.6 (Theorem I.12.5 [22])**.** *Let K be a group with the maximal and minimal conditions for normal subgroups.*

(a) *If $K = K_1 \times \cdots \times K_r = L_1 \times \cdots L_s$ are two direct decompositions of K into directly indecomposable non-trivial subgroups L_i and K_j then $r = s$ and there exists a re-numbering of the subgroups L_i so that $L_i \cong K_i$ for all i.*

(b) *The direct decomposition $K = K_1 \times \cdots \times K_r$ into directly indecomposable factors K_i is unique (up to permutation) if and only if for $i \neq j$ the only homomorphism from K_i into $Z(K_j)$ is the trivial homomorphism.*

Proposition 3.1.7. *Let $G_1, G_2, \ldots, G_n$ be directly indecomposable non-isomorphic centerless finite groups for some positive integer n. The automorphism group of $G_1 \times G_2 \times \cdots \times G_n$ is isomorphic to Aut $G_1 \times$ Aut $G_2 \times \cdots \times$ Aut G_n.*

Corollary 3.1.8. *Let $G_1, G_2, \ldots, G_n$ be directly indecomposable non-isomorphic groups for some positive integer n. Then $G_1 \times G_2 \times \cdots \times G_n$ is complete if and only if G_i is complete for each i with $1 \leq i \leq n$.*

Proposition 3.1.7 follows easily from Theorem 3.1.6. The factors have trivial centers and are non-isomorphic, meaning that the automorphisms of the direct product must preserve each component.

The corollary provides a convenient way to construct complete groups. One direction of the biconditional in Corollary 3.1.8 is the direct result of Proposition 3.1.7. However, if a direct product is complete then each factor, being a normal subgroup, is characteristic. The remainder of the corollary follows easily.

Now that we have many examples of complete groups with which to work, one begins to wonder what is the importance of complete groups. The construction of the *automorphism tower of a centerless group* implies that complete groups actually play an important role in the study of groups. Let G be a group with trivial center. To construct the automorphism tower of G, let $G_0 = \text{Inn } G \cong G$ and $G_1 = \text{Aut } G$.

Proposition 3.1.9 (Wielandt (42) [39])**.** *Suppose that* G *is a finite group with* $Z(G) = 1$. *Then* $C_{\mathit{Aut}\ G}(\mathit{Inn}\ G) = 1$.

Proposition 3.1.9 then tells us that G_1 also has a trivial center. We may therefore continue the construction by letting G_2 be the automorphism group of G_1. Continuing in this manner we obtain a sequence of finite groups,

$$G = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots ,$$

and this sequence is called the automorphism tower of G. From the construction one is inspired to ask, is this sequence strictly increasing? If not, then there is a group, say G_i, such that

$$G_{i+1} = \text{Aut } G_i = G_i.$$

Since all of the groups in the sequence have a trivial center, this means that G_i is complete. Moreover if G_i is complete for some i then the automorphism tower would in fact stabilize. We refer to the *height* of the automorphism tower of G as the minimal integer i such that G_i is complete.

Notice that if a centerless group G has an automorphism tower of finite height then G can be embedded as a subnormal subgroup in a complete group. In 1938 H. Wielandt also proved the following remarkable theorem.

Theorem 3.1.10 (Wielandt (45) [39])**.** *Suppose that* G *is a non-trivial finite group with* $Z(G) = 1$. *The automorphism tower of* G *is of finite height. (In particular,* G *can be embedded subnormally in a finite complete group.)*

The proof even places a rough bound on the height of the automorphism tower, though more recent works have not only improved this bound but extended the discussion to infinite groups and groups with a non-trivial center (see [35], [36], [14], or [15] for more details). The proof of Theorem 3.1.10 and the supporting results can be found in Chapter 1 Section 6 of [27], as well as in the Appendices of [40]. This result was further broadened a few decades later, using wreath products, to give the following.

Theorem 3.1.11 (Rose [30]). *Every finite group can be embedded subnormally in a finite complete group. Moreover, a finite solvable group can be embedded subnormally in a finite solvable complete group.*

Just as Cayley provided inspiration to study symmetric groups by proving that every group can be embedded in one, Wielandt and Rose together provided the main motivation to understand complete groups.

Recent studies of finite complete groups have focused on characterizing groups of a given type. These studies have culminated in the characterization of complete metabelian groups and complete metanilpotent groups. For the latter result, recall that a Carter subgroup of a finite group is a subgroup that is both nilpotent and self-normalizing.

Theorem 3.1.12 (Theorem 1 [12]). *Let G be a metabelian group. Then Out $G = 1$ if and only if $|G| \leq 2$ or G is the direct product of holomorphs of cyclic groups of distinct odd primary orders.*

Theorem 3.1.13 (Theorem 1 [29]). *Let G be a metanilpotent group and let N be the Fitting subgroup of G. Then G is complete if and only if $Q = G/N$ is a Carter subgroup of Out N and Q acts without fixed points on $Z(N)$.*

With regards to the former characterization, recall from Burnside's original definition of complete that if G has a trivial center and Out $G = 1$ then G is complete. Thus Theorem 3.1.12 tells us that the only complete metabelian groups are those that are a direct product of holomorphs of cyclic groups of distinct odd primary orders. We already knew that groups of these types were complete, from Proposition 3.1.5 and Corollary 3.1.8. However, the interesting consequence of Theorem 3.1.12 is that there is no other way to construct a complete metabelian group.

Due to the difficulty in calculating automorphism groups and identifying their subgroups, it can be difficult to apply Theorem 3.1.13. However, in the case of abelian-by-nilpotent groups T. M. Gagen determined the relevant automorphism groups and their

Carter subgroups, deriving the following classification of complete abelian-by-nilpotent groups.

Theorem 3.1.14 (Theorem 0 [11]). *Let G be an abelian-by-nilpotent group. Then Out $G = 1$ if and only if $|G| \leq 2$ or G is the direct product of groups of the form $A_i \rtimes X_i$ where A_i is a homocyclic p-group of odd-order, its complement X_i is the normalizer of a Sylow 2-subgroup of Aut A_i, and for $i \neq j$ the groups $A_i \neq A_j$.*

Moreover, B. Hartley and D. J. S. Robinson further examined complete metanilpotent groups and determined a characterization from another perspective, with interesting consequences. A group G is called p-dominated if it is a complete group whose Fitting subgroup is a Sylow p-subgroup with a nilpotent quotient.

Theorem 3.1.15 (Theorem 2 [16]). *A metanilpotent group is complete if and only if it is a direct product of p-dominated groups for different primes p.*

Theorem 3.1.16 (Corollary 3 [16]). *A complete metanilpotent group splits over its Fitting subgroup.*

Though these results were published by three authors and at varying times, beneath the surface of the proofs is an interesting result that B. Hartley and D. J. S. Robinson published in 1980.

Proposition 3.1.17 (Lemma 2 [16]). *Let N be a normal subgroup of a group K such that $C_K(N) = Z(N) = A$. Assume that $Q = K/N$ is nilpotent and regard Q as a subgroup of Out N. If K is complete then*

(i) *Q is self-normalizing as a subgroup of Out N, and*

(ii) *Q acts on A without fixed points.*

Lemma 3.1.17 is straightforward to prove using cohomology of groups. In fact, this proposition inspired, in part, our approach to characterizing complete groups in Section 3.3. A thorough discussion of these results can be found in [29].

Another area of focus in the recent studies of complete groups has been in building examples. All of the examples of complete groups that we discussed have even order, and these were the principal examples for many years. Whether odd-order complete groups existed became a long standing question, which was answered the 1970's. The construction by M. V. Khoroshevskii [23] for a family of odd-order complete groups appeared in print in 1974. These examples were π-groups, where π was a set of at least 6 primes. R. Dark [5] exhibited a much smaller complete group of order $3 \cdot 19 \cdot 7^{12}$, which appeared in print in 1975.

B. Schuhmann published an article [32], based on her dissertation research, determining that there cannot exist a finite solvable complete group of odd-order with solvable length less than 4. Moreover, she described odd-order solvable complete groups of solvable length 4.

Theorem 3.1.18 (Theorem 2 [32]). *Let G be a solvable complete group of odd-order and chief series length 4. Then $G = N \rtimes H$ where N is a p-group and H is a group of order q for primes $p \neq q$.*

Schuhmann's paper was the inspiration for our research in Section 3.2 on odd-order complete groups of solvable length at least 5.

Also in 1984 H. Heineken, Schuhmann's dissertation advisor, published with J. C. Lennox an article [18] in which it was determined that finite solvable complete groups cannot be embedded subnormally in the derived subgroup of another finite group. The paper also provides an example demonstrating that subnormal complete subgroups may have arbitrarily large defect. More recent work by Heineken [19] and P. Soules [33], also a student of Heineken, have provided methods for constructing complete groups of small odd-order.

3.2 Odd-order Complete Solvable Groups

In this section we will assume that the reader is familiar with some results and definitions from the study of solvable groups. Standard references for this area of group theory include [22], although [37] provides a nice introduction to the topic as well. Throughout this section we make use of G. Frattini's Argument, Theorem 3.2.1. We also often employ the Schur-Zassenhaus Theorem (Theorem 3.2.2) which follows.

Theorem 3.2.1 (Frattini's Argument). *[9, Proposition 6.6] Let G be a finite group, let H be a normal subgroup of G, and let P be a Sylow p-subgroup of H. Then $G = HN_G(P)$ and $|G : H|$ divides $|N_G(P)|$.*

Theorem 3.2.2 (Schur's Theorem)**.** *[9, Theorem 17.39] If N is a normal Hall π-subgroup of G then there exists a complement to N in G.*

Recall the definitions regarding subgroup series of a group G. We remind readers of the earlier definition of a solvable group.

Definition 3.2.3. Let G be a group.

1. A series of subgroups, $\{G_i\}$, where

   $$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_i \trianglelefteq \cdots \trianglelefteq G_n = G$$

   is called a *composition series* of G if there is no subgroup H such that $G_i \lhd H \lhd G_{i+1}$.

2. The *length of the composition series* $\{G_i\}$ is the integer n.

3. A series of subgroups, $\{G_i\}$, where

   $$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_i \trianglelefteq \cdots \trianglelefteq G_n = G$$

   is called a *chief series* of G if $G_i \trianglelefteq G$ for each i and there does not exist $H \trianglelefteq G$ with $G_i < H < G_{i+1}$. The quotients of a chief series are known as *chief factors.*

4. Recursively define the *derived series of* G by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

5. The group G is *solvable* if there exists a positive integer i such that $G^{(i)} = 1$. The *derived, or solvable, length of* G is the smallest integer i such that $G^{(i)} = 1$.

Every finite group has a composition series, since a finite group has finitely many subgroups. So every finite group has a chief series, such as the series

$$1 \trianglelefteq A_5 \trianglelefteq S_5$$

in S_5. In a solvable group the chief factors are elementary abelian p-groups, though the prime p may differ for different factors. Hence a minimal normal subgroup of a solvable group is an elementary abelian p-group for a prime p. Moreover it is known that the index $|G : G_{n-1}|$ is prime in a solvable G. (See [37] or [22] for details.)

Theorem 3.2.4 (The Jordan-Hölder Theorem)**.** *[37, Theorem 1.2] Let G be a group. If $\{G_i\}$ is a composition series of G with length r and $\{H_j\}$ is a composition series of G with length s then $r = s$ and there is a permutation $\sigma \in S_r$ so that the groups G_i/G_{i-1} and $H_{i\sigma}/H_{i\sigma-1}$ are isomorphic for all i.*

Theorem 3.2.4 implies that every chief series of a finite group has the same length. By the definitions alone, the derived series of G is not necessarily a chief series. Even if G is solvable and the derived series terminates with 1, there could be normal subgroups $H \trianglelefteq G$ such that $G^{(i+1)} < H < G^{(i)}$. Hence the solvable length of a solvable group G is less than or equal to the length of a chief series.

In a 1984 article B. Schuhmann showed that if G is a solvable odd-order complete group then it must have solvable length at least 4 [32]. In this section we will consider the structure of a complete odd-order group of solvable length n for an integer $n \geq 5$, showing that at the bottom of the solvable length of such a group is somewhat related to the number of primes dividing its order.

In Schuhmann's original paper many of her proofs use an initial lemma, which is re-stated below. We, too, utilize this lemma in our proofs.

Lemma 3.2.5 (Lemma 1 [32]). *Let G be a solvable group with a minimal normal subgroup N such that (i) there is a complement H of N in G and (ii) both N and $Z(H)N/N$ are of odd order. Then G is not complete.*

Schuhmann's proof does not require that N be a minimal normal subgroup, instead it only uses the fact that N is abelian. We present the generalization as the next result, and give Schuhmann's proof.

Lemma 3.2.6. *Let G be a solvable group with a non-trivial abelian normal subgroup N such that (i) there is a complement H of N in G, (ii) N has odd order, and (iii) either $Z(H)$ or $C_H(N)$ is of odd order. Then G is not complete.*

Proof. Since N is abelian and of odd order, the map $\varphi : N \to N$ defined by $\varphi(n) = n^{-1}$ is a non-trivial automorphism of N. Notice that

$$\varphi(n^h) = (n^h)^{-1} = (n^{-1})^h = \varphi(n)^h$$

so Equation 1.1 from Chapter 1.2 is satisfied. Therefore φ can be extended to an automorphism φ^* of G by $\varphi^*(nh) = \varphi(n)h = n^{-1}h$.

Suppose that φ^* is an inner automorphism of G. Since $|N|$ is odd we know that φ^* is not equal to τ_m for any $m \in N$. Thus $\varphi^* = \tau_{mk}$ for $k \neq 1$. Yet $\varphi^*(h) = h$ for all $h \in H$ and therefore $mk \in C_G(H)$, in particular $k \in \mathrm{Z}(H)$. Also $|\varphi^*| = 2$ so $|mk| = 2$. Therefore $mk\ mk = mm^{k^{-1}}k^2 = 1$, and since G is a semidirect product we therefore have that $mm^{k^{-1}} = k^2 = 1$. In other words, k has order 2 and centralizes N. However at least one of $|\mathrm{Z}(H)|$ or $|C_H(N)|$ is odd so no such k can exist. Thus the extension φ^* cannot be induced by conjugation with an element of G.

Hence φ^* is a non-inner automorphism and G is not complete. ■

With these facts we will set our notation. Let G be a complete group of odd-order with solvable length n, where $n \geq 5$. Since G is solvable the index of a maximal normal subgroup is a prime and the indices in a chief series are powers of primes, though perhaps not pairwise coprime. Let p_i be an odd prime and α_i be a positive integer for $i = 1, 2, \ldots, n$. Fix the following as a chief series of G,

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{i-1} \trianglelefteq G_i \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G, \tag{3.1}$$

where $|G_i : G_{i-1}| = p_i^{\alpha_i}$ for each i. Note that $\alpha_n = 1$. Moreover, the solvability of G yields that the quotient G_i/G_{i-1} is elementary abelian.

Our first result connects the number of primes dividing the order of G and the number of distinct primes p for which G has a minimal normal subgroup of p-power order.

Proposition 3.2.7. *In the chief series 3.1, the minimal normal subgroup G_1 is not a Sylow p_1-subgroup of G. Specifically, in any chief series of a complete solvable odd-order group G there is a chief factor whose order is a power of the same prime as the order of the minimal normal subgroup in that fixed chief series.*

Proof. Suppose G_1 were a Sylow p_1-subgroup of G. Since G_1 is normal, G must split over p_1 by the Schur-Zassenhaus theorem. Then G_1 would satisfy the conditions of Lemma 3.2.6 and G would not be complete. Therefore G_1 is not a Sylow p_1-subgroup of G. ■

In other words, if G has a minimal normal subgroup of order p^α for some prime p then in our fixed chief series there exists at least two distinct integer, i and j, such that $p = p_i = p_j$. This is true for every minimal normal subgroup of G. From this we see:

Corollary 3.2.8. *If there exists m distinct primes p for which there exists a minimal normal subgroup of G of p-power order then $n \geq 2m$.*

Moreover, if $p_1 \neq p_2$ then Lemma 3.2.7 lets us further explore the chief series of G.

Lemma 3.2.9. *If, in the chief series 3.1, we have that* $p_1 \neq p_2$ *then there exists a minimal normal subgroup,* Q*, of* G *with* $|Q| = p_2^{\alpha_2}$ *such that* $G_2 = G_1 \times Q$*.*

Proof. Let $p_1 \neq p_2$. Then G_1 is a normal Sylow p_1-subgroup of G_2. By the Schur-Zassenhaus Theorem G_1 has a complement Q in G_2. Since G_2 is a $\{p_1, p_2\}$-group we know that Q must be a Sylow p_2-subgroup of G_2. By Frattini's Argument the normality of G_2 implies that

$$G = N_G(Q)G_2 = N_G(Q)QG_1 = N_G(Q)G_1.$$

Let $X = N_G(Q) \cap G_1$. This subgroup is normal in $N_G(Q)$ since $G_1 \trianglelefteq G$; hence $N_G(Q) \leq N_G(X)$. Yet G_1 is abelian so $X \trianglelefteq G_1$. Then $X \trianglelefteq G$. As a subgroup of a minimal normal subgroup, either $X = 1$ or $X = G_1$.

In the first case G_1 would have a complement in G, so by Lemma 3.2.6 the group G would not be complete. Therefore $X = G_1$ and G_1 normalizes Q. Then $Q \trianglelefteq G$ and $G_2 = G_1 \times Q$. Yet $Q \cong G_2/G_1$ is an elementary abelian p_2-group. If Q were not a minimal normal subgroup of G, then there would be a chief series of length at least $n + 1$. Thus Q must be a minimal normal subgroup of G. ■

A solvable group always has minimal normal subgroups, however there is no promise that there is more than one or that for a given prime p dividing the order of G there will actually be a minimal normal subgroup of p-power order. Hence Lemma 3.2.9 gives us extra strength in the case of a solvable complete group.

In Lemma 3.2.9 we supposed that the bottom two primes, p_1 and p_2, of our chief series were distinct. We now consider the case where the bottom k primes of the chief series are distinct, and the result is similar.

Proposition 3.2.10. *Let* G_k *be a subgroup in the chief series 3.1 such that the indices* $p_i^{\alpha_i}$ *for* $i \leq k$ *are pairwise coprime. Then* G_k *is a direct product of its elementary abelian Sylow subgroups, each of which is a minimal normal subgroup of* G*.*

Proof. We proceed by induction on k. The statement is clearly true for $k = 1$, when G_k is a minimal normal subgroup of G. Lemma 3.2.9 established the case for when $k = 2$.

Assume the hypothesis holds for $k - 1$, so that G_{k-1} can be decomposed as

$$G_{k-1} = P_1 \times P_2 \times \cdots \times P_{k-1}$$

where each P_i is a Sylow p_i-subgroup of G_k and a minimal normal subgroup of G. Each P_i is an elementary abelian group, therefore G_{k-1} is abelian. Moreover, since G_{k-1} is a normal Hall subgroup of G_k, there exists a complement P_k of G_{k-1} in G_k. Our goal is to show that P_k is a minimal normal subgroup of G and that $G_k = G_{k-1} \times P_k$.

By Frattini's Argument

$$G = N_G(P_k)G_k = N_G(P_k)P_kG_{k-1} = N_G(P_k)G_{k-1}.$$

Let $X = G_{k-1} \cap N_G(P_k)$. Since G_{k-1} is abelian, G_{k-1} normalizes X. However X is also normalized by $N_G(P_k)$ so $X \trianglelefteq G$. Since $X \leq G_{k-1}$ there are three possible cases.

First, suppose that $X = 1$. Then G_{k-1} satisfies the hypotheses of Lemma 3.2.6, which implies that G is not complete. Thus this case cannot occur. Suppose instead that X is properly contained in G_{k-1}. Yet G_{k-1} can be decomposed as a direct product of directly indecomposable minimal normal subgroups of G. By Theorem 3.1.6 we know this decomposition of G_{k-1} is unique, hence the normal subgroup X is a direct product of some of the factors of G_{k-1}. We can then decompose G_{k-1} as $X \times Y$ where X is some number of the minimal normal subgroups in our original decomposition of G_{k-1} and Y is the remaining factors. Since Y is generated by abelian normal subgroups of G we have that Y is an abelian normal subgroup of G. We can further reduce the expression of G obtained using Frattini's Argument. The result is that

$$G = N_G(P_k)G_{k-1} = N_G(P_k)(X \times Y) = N_G(P_k)Y$$

where $Y \cap N_G(P_k) = 1$. Now Y would satisfy the requirements of Lemma 3.2.6, yielding that G is not complete. This contradiction forces $X = G_{k-1}$ as the only possible case.

Since $X = G_{k-1}$ we know that $P_k \trianglelefteq G$. If P_k were not a minimal normal subgroup of G then there would exist a chief series of G of length greater than n. Hence P_k must be a minimal normal subgroup of G and $G_k = G_{k-1} \times P_k$ as desired. ■

In a typical solvable group, each chief factor is elementary abelian but the subgroup for which the chief factor is a quotient need not be abelian. This is still the case for complete solvable groups but Proposition 3.2.10 allows for the possibility of a large abelian group in the lower end of a chief series.

We summarize the main results of this section in the following theorem.

Theorem 3.2.11. *Let k be the maximal integer such that in the chief series 3.1 the indices of the chief factors $p_i^{\alpha_i}$ where $i \leq k$ are pairwise coprime.*

(a) Then n is at least $2k$.

(b) There exists a chief series of G in which the bottom two indices are both powers of p_{k+1}.

Proof. Since Proposition 3.2.10 gave us that G_k is a direct product of k minimal normal subgroups of G, we can develop a chief series of G with any of these factors of G_k on the bottom. Thus Proposition 3.2.7 insists that each prime dividing $|G_k|$ reappear in the chief series above G_k. Yet $|G_k|$ has k distinct prime factors, so $2k \leq n$. This establishes part (a).

For part (b) let us write G_k as

$$G_k = P_1 \times P_2 \times \cdots \times P_k$$

where P_i is a Sylow p_i-subgroup of G_k and a minimal normal subgroup of G. We know that $p_{k+1} = p_i$ for some $i \leq k$, so assume without loss of generality that $p_k = p_{k+1}$. Then G_{k-1} is a normal Hall subgroup of G_{k+1} and in fact

$$G_{k-1} = P_1 \times P_2 \times \cdots \times P_{k-1}.$$

By the Schur-Zassenhaus Theorem there exists a complement H of G_{k-1} in G_{k+1}. Moreover, H is a Sylow p_k-subgroup of G_{k+1}. We see then that

$$G = N_G(H)G_{k+1} = N_G(H)G_{k-1}$$

by the Frattini argument. Let $X = N_G(H) \cap G_{k-1}$. Both $N_G(H)$ and G_{k-1} normalize X, making it normal in all of G.

If X were trivial then G_{k-1} would satisfy the requirements of Lemma 3.2.6 and force G to no longer be complete. Suppose instead that X is properly contained in G_{k-1}. Since each component of G_{k-1} is a minimal normal subgroup, if X intersects a component non-trivially then it must contain the entire factor. Let Y be the direct product of factors that intersect X trivially. We can decompose G_{k-1} as $G_{k-1} = X \times Y$ and, since Y is generated by normal abelian subgroups of G, we have that Y itself if a normal abelian subgroup of G. Moreover $Y \cap N_G(H) = 1$, so by Lemma 3.2.6 the group G would not be complete.

Therefore the only possibility is that $X = G_{k-1}$ and hence $H \trianglelefteq G$. Additionally $H \cap G_k = P_k$ is a minimal normal subgroup of G. We can then define a second chief series of G by the following.

$$\widetilde{G}_i = \begin{cases} G_i & \text{if } n \geq i \geq k+1 \\ H \times P_1 \times \cdots \times P_{i-2} & \text{if } k \geq i \geq 3 \\ H & \text{if } i = 2 \\ P_k & \text{if } i = 1 \end{cases}$$

The new chief series is exactly the same as the original for $i \geq k+1$. However the initial terms in the series have been rearranged so that the first two terms are P_k and then H. This gives the desired result: a chief series in which the bottom two indices are powers of the same prime, p_k. ■

We see that for a complete solvable odd-order G with k minimal normal subgroups of distinct primary orders, then the solvable length of G must be at least $2k$. Furthermore if M is a minimal normal subgroup with p-power order of a complete odd-order group G then there must be at least two factors of p dividing $|G|$.

Our results pertain to complete odd-order groups with a chief series wherein the lower indices are coprime. The examples of complete odd-order groups mentioned in Section 3.1 are constructed by taking a non-abelian p-group for a prime p and extending that group. Therefore the results presented in this section do not necessarily apply to the known examples of complete odd-order groups.

3.3 Cohomology and Classification Theorems

In this section we will use the facts from Section 1.3 and new results to classify complete wreath products and apply that classification to standard wreath products. Additionally, we will classify complete Frobenius groups.

The proofs of these classifications rely on the same outline, despite the differences in the group structures. We attempt to highlight the approach and so will first apply it to an easy example in Subsection 3.3.1. Before we can continue we must discuss the next fact, which follows from the Lattice Isomorphism Theorem [9, Theorem 3.3.20].

Proposition 3.3.1. *Let K be any finite group with $N \trianglelefteq K$ and $U \leq K$. Then UN is self-normalizing in K if and only if UN/N is self-normalizing in K/N.*

Proof. To simplify the notation we place a bar over a symbol to represent its image in K/N. Suppose that UN is self-normalizing in K and let $k \in K$ be such that $\overline{k}$ normalizes $\overline{U}$. Then for every $u \in U$ there exists $u_1 \in U$ and $n_1 \in N$ such that $u^k = u_1 n_1 \in UN$. Thus, since $N \trianglelefteq K$, if $u \in U$ and $n \in N$ then there exists $u_1 \in U$ and $n_1, n_2 \in N$ such that

$$(un)^k = u^k n^k = u_1 n_1 n_2 \in UN.$$

Hence $k \in N_K(UN)$. Since UN is self-normalizing, then $k \in UN$. Therefore $\overline{k} \in \overline{U}$.

Now suppose that $\overline{U}$ is self-normalizing in $\overline{K}$ and let $k \in K$ normalize UN. If $\overline{u} \in \overline{U}$ note that there exists $u_1 n_1 \in UN$ such that

$$\overline{u}^{\overline{k}} = \overline{u^k} = \overline{u_1 n_1} = \overline{u_1}.$$

Thus $\overline{k}$ normalizes $\overline{U}$, and so $\overline{k} \in \overline{U}$. Then k is in the pre-image of $\overline{U}$, meaning that $k \in UN$. Hence UN is self-normalizing. ∎

Proposition 3.3.1 is an elementary fact that plays a role in several parts of this chapter is used implicitly in many of the results we discussed in Section 3.1. We use Proposition 3.3.1 in the case of a semidirect product, where U is the image of the complement in the automorphism group of the base (playing the role of K).

We present now a result demonstrating how extending automorphisms of normal subgroups can aid in the study of complete groups. This observation is so easy to prove that surely it is well known among those who study complete groups, although we have yet to find its explicit statement.

Lemma 3.3.2. *Let $G = N \rtimes H$ and suppose that H acts faithfully on N by conjugation. Suppose further that G is complete. If $\varphi \in Aut\ N$ that centralizes those automorphisms of N induced by H then $|\varphi| \mid |G|$.*

Proof. Suppose that H acts faithfully on N by conjugation in $G = N \rtimes H$, a complete group. Let φ be an automorphism of N that commutes with those automorphism induced by H. Then define an automorphism φ^* of G by

$$\varphi^*(nh) = \varphi(n)h.$$

Since φ commutes with H it's easy to see that φ^* is in fact an automorphism of G.

As G is complete there must exist $g \in G$ such that $\varphi^* = \tau_g$ and hence $|\varphi^*| \mid |G|$. Yet since $\varphi^*_{|N} = \varphi$ we have that $|\varphi| \mid |\varphi^*|$. Thus $|\varphi| \mid |G|$. ■

Lemma 3.3.2 is a quaint result; most of the results regarding complete groups are not related so directly to the order of the group. Yet the proof illustrates how inducing automorphisms from normal subgroups can be useful.

In the next few sections we will present an outline of our argument in the form of an example, and then employ that argument to characterize complete permutational wreath products. Once that characterization is established we use it to understand complete standard wreath products and some applications. Lastly we will characterize complete Frobenius groups.

3.3.1 An Example: $[C_3 \times C_3]\mathbf{GL}_2(3)$

Let $G = [C_3 \times C_3]\mathrm{GL}_2(3)$. Mills [25, Theorem 1] established that the holomorph of any odd-order abelian group is complete, so we already know that G is a complete group. The aim of this example is to illustrate the application of Propositions 1.3.2 and 1.3.6, which is roughly the same approach we later take to classify complete wreath products and complete Frobenius groups.

An outline of the approach is: First identify a subgroup that is characteristic and fits the criteria of either Proposition 1.3.2 or Proposition 1.3.6. Then restrict an automorphism of the group to this subgroup. Use our propositions to force the automorphism to be inner. We will actually perform this approach twice with G – first with characteristic subgroup $C_3 \times C_3$ and Proposition 1.3.6, then with a second characteristic subgroup and Proposition 1.3.2. We begin with a lemma describing the structure of G.

Lemma 3.3.3. *Let $G = [C_3 \times C_3]GL_2(3)$. Let J be the subgroup of G formed by the semidirect product $[C_3 \times C_3]Q$ where $Q = \mathbb{O}_2(GL_2(3))$ is isomorphic to the group of quaternions, Q_8. The following facts hold.*

(a) $Z(G) = 1$

(b) $C_G(C_3 \times C_3) = C_J(C_3 \times C_3) = C_3 \times C_3$

(c) $C_3 \times C_3$ *char* G

(d) J *char* G

(e) $C_G(J) = 1$

(f) $H^1(Q, C_3 \times C_3) = 1$

Proof. Since G is the holomorph of $C_3 \times C_3$, we know that $\mathrm{GL}_2(3)$ acts faithfully on $C_3 \times C_3$. This results in both part (a) and $C_G(C_3 \times C_3) = C_3 \times C_3$. Yet the rest of part (b) follows easily from this fact, too.

Notice that $C_3 \times C_3$ is the largest normal 3-subgroup of G, hence $\mathbb{O}_3(G) = C_3 \times C_3$ and this subgroup is characteristic. Also J is the largest normal $2,3$-subgroup of G hence J is also characteristic.

As Q acts faithfully on $C_3 \times C_3$ we know that $\mathrm{Z}(J) = 1$. This combined with part (b) tell us that $C_G(J) = 1$. Finally, $|Q|$ and $|C_3 \times C_3|$ are relatively prime and that implies part (f) by Lemma 1.3.7. ■

We will continue to use the notation established for G in the above lemma, and can now prove the desired result of the section.

Proposition 3.3.4. *The group* $[C_3 \times C_3]GL_2(3)$ *is complete.*

Proof. The previous lemma established that $\mathrm{Z}(G) = 1$. We need only show that every automorphism of G is inner. Let φ be an automorphism of G. The restriction of φ to $C_3 \times C_3$ must be an element of $\mathrm{Aut}\ C_3 \times C_3 = \mathrm{GL}_2(3)$. Therefore up to multiplication by an inner automorphism of G we have that φ fixes $C_3 \times C_3$ point-wise. Let $g \in G$ be the element such that $\varphi_0 = \varphi(\tau_g)^{-1}$ is the automorphism that fixes $C_3 \times C_3$ point-wise.

Let $x \in C_3 \times C_3$ and $y \in J$. Then

$$x^{\varphi_0(y)} = \varphi_0(y^{-1}xy) = \varphi_0(x^y) = x^y.$$

So $y\varphi_0(y)^{-1}$ centralizes all of $C_3 \times C_3$. Yet J is characteristic, implying that this product is still an element of J. Lemma 3.3.3(b) tells us that if we pass to the quotient $J/(C_3 \times C_3)$ then φ_0 is trivial.

As an automorphism of J that fixes both $C_3 \times C_3$ and $J/C_3 \times C_3$ point-wise, Corollary 1.3.5 says that φ_0 corresponds to an element in $H^1(Q, C_3 \times C_3)$ and hence $(\varphi_0)_{|J}$ is an inner automorphism of J.

Now let $j \in J$ be the element where $\varphi_1 = \varphi_0(\tau_j)^{-1}$ is trivial on J. Proposition 1.3.2 may be used, given Lemma 3.3.3, so we have that φ_1 is trivial on all of G.

Unraveling the substitutions, we have that $\varphi(\tau_g)^{-1}(\tau_j)^{-1}$ is trivial on all of G for some $g \in G$ and some $j \in J$. Therefore φ is an inner automorphism of G and G is a complete group. ■

The example with G explicitly demonstrates how both Propositions 1.3.2 and 1.3.6 are applied. The remaining subsections will heavily rely on this same type of argument to prove that a group is complete.

3.3.2 Complete Permutational Wreath Products

In this subsection we will classify complete permutational wreath products, maintaining the same notation as used in Chapter 2. Throughout this section let G be a finite group and H be a finite permutation group of degree n. Let B denote the base of $G \wr H$. The ultimate goal of this section is to show that a (permutational) wreath product $G \wr H$ is complete if and only if the following three properties hold:

(a) The group G is complete.
(b) The base of $G \wr H$ is a characteristic subgroup.
(c) The image of H in the outer automorphism group of the base is self-normalizing.

It is not surprising when seeing these properties that we investigated the conditions under which the base of a permutational wreath product is characteristic in Chapter 2. We also see a similarity between these properties and the conditions for a complete Frobenius group later in the chapter.

Our first result of the subsection will remind readers of a few facts about wreath products that were not discussed in Chapter 2.

Proposition 3.3.5. *Suppose G is a centerless group and H is a permutation group of degree n. Let B be the base of the wreath product $G \wr H$. Then:*

(1) $Z(B) = 1$,
(2) $C_{G \wr H}(B) = 1$,
(3) $Z(G \wr H) = 1$,
(4) $H^m((G \wr H)/B, Z(B)) = 1$ *for all positive integers m, and*

(5) *there is an injective homomorphism* $\rho : G \wr H \to \mathit{Aut}\ B$ *that takes an element of the wreath product to the automorphism of* B *it induces via conjugation in* $G \wr H$*.*

Proof. From the hypothesis it is clear that (1) holds. As the permutation representation of H is faithful, no element of H commutes with every element of the base. This fact and (1) imply $C_{G\wr H}(B) = 1$, establishing (2) and (3).

In light of (1) and Lemma 1.3.8, we see that the cohomology group $H^m((G \wr H)/B, \mathrm{Z}(B))$ is trivial for all positive integers m. Turning to (5), we require use of Proposition 1.3.1. Since $B \trianglelefteq G \wr H$ and the centralizer of B is trivial, Proposition 1.3.1 gives that $G \wr H$ can be embedded in Aut B as described. ■

With the structure of $G \wr H$ established, we can now describe how to induce automorphisms of G up to $G \wr H$.

Lemma 3.3.6. *If* G *is not complete then* $G \wr H$ *is not complete.*

Proof. In a wreath product the diagonal subgroup of B always commutes with H. Suppose that G is not complete because its center is non-trivial and let $z \in \mathrm{Z}(G)$. Then $f = (z, z, \ldots, z) \in \mathrm{Z}(B)$ but f also centralizes H. Hence the center of $G \wr H$ is non-trivial and $G \wr H$ is not complete.

Suppose instead that $\mathrm{Z}(G) = 1$ but that G has non-trivial outer automorphisms, with φ one such automorphism. Naturally extend the definition of φ to B by letting φ act on each coordinate as described below where $b \in B$.

$$b^{\varphi}(i) = (b(i))^{\varphi}$$

To simplify the notation we use the same symbol to represent φ as an automorphism of G and as an automorphism of B. Now define $\varphi^* : G \wr H \to G \wr H$ as follows where $b \in B$ and $h \in H$:

$$(bh)^{\varphi^*} = b^{\varphi}h.$$

Let $b \in B$ and $h \in H$. The next argument shows that conjugation by h commutes with φ on B.

$$\begin{aligned} b^{h\varphi}(i) &= (b^h(i))^{\varphi} \\ &= (b(ih^{-1}))^{\varphi} \\ &= b^{\varphi}(ih^{-1}) \\ &= b^{\varphi h}(i) \end{aligned}$$

Since h only permutes the coordinates and φ is being applied individually to each coordinate, the permutation can occur before or after the application of φ with no difference in effect. It is also straightforward to check that φ^* satisfies the definition of a homomorphism. Let $b_1, b_2 \in B$ and $h_1, h_2 \in H$. Then note:

$$\begin{aligned}
(b_1 h_1 b_2 h_2)^{\varphi^*} &= (b_1 b_2^{h_1^{-1}} h_1 h_2)^{\varphi^*} \\
&= (b_1 b_2^{h_1^{-1}})^{\varphi} h_1 h_2 \\
&= b_1^{\varphi} b_2^{h_1^{-1}\varphi} h_1 h_2 \\
&= b_1^{\varphi} b_2^{\varphi h_1^{-1}} h_1 h_2 \\
&= b_1^{\varphi} h_1 b_2^{\varphi} h_2 \\
&= (b_1 h_1)^{\varphi^*} (b_2 h_2)^{\varphi^*}.
\end{aligned}$$

For $b \in B$ and $h \in H$, the element bh is in the kernel of φ^* if and only if $b^{\varphi} h = 1$. This requires that $b^{\varphi} = h^{-1}$. Since $B \cap H = 1$ the kernel of φ^* is trivial. Thus φ^* is an automorphism of $G \wr H$ and we proceed to prove that φ^* is not an inner automorphism.

Suppose by way of contradiction that φ^* is inner. As the elements of H permute the coordinate entries and we are going to consider the effect of φ^* on a diagonal element, we may reduce to the case where $\varphi^* = \tau_b$ for some $b \in B$. Suppose that $b(i) = g_i$ where $g_i \in G$ for each i. Since the center of of G is trivial there exists $g \in G$ such g does not commute with at least one of the g_i. Consider the effect of φ versus that of τ_b on the diagonal function f with $f(i) = g$ for all i:

$$f^b(i) = g^{g_i} \text{ and } f^{\varphi^*}(i) = g^{\varphi}.$$

Notice that f^{φ^*} is still a diagonal element, meaning that if $f^{\varphi^*} = f^b$ then $g_i = g_j$ for all i and j. Then φ would be an inner automorphism of G, contrary to the hypothesis. Thus if φ^* is induced from a non-inner automorphism of G then φ^* is a non-inner automorphism of $G \wr H$. Therefore if G is not complete then neither is $G \wr H$. ■

We are now ready to prove the characterization of complete wreath products.

Theorem 3.3.7. *Let G be a finite group, let H be a faithful permutation group of finite degree n, and let B represent the base of $G \wr H$. Then $G \wr H$ is complete if and only if*

- **(a)** *G is complete,*
- **(b)** *the base B is characteristic in $G \wr H$, and*
- **(c)** *H is self-normalizing as a subgroup of Out B.*

Proof. Let $G \wr H$ be complete. Since $B \trianglelefteq G \wr H$, the base must be characteristic and (b) holds. According to Proposition 3.3.6 the group G is complete, thus establishing (a) and allowing the use of Proposition 3.3.5. We need to show that the image of H in Out B is self-normalizing. Since $C_{G \wr H}(B) = 1$ we know that D, the image of $G \wr H$ in Aut B, is isomorphic to $G \wr H$ and $\overline{D} = D/\text{Inn } B$ is isomorphic to H.

To finish the forward direction of the proof we'll prove that D is self-normalizing, which then implies that $\overline{D}$ is self-normalizing by Proposition 3.3.1 with $N = \text{Inn } B$ and U the image of H in Aut B. Let φ be an automorphism of B that normalizes D. Proposition 3.3.5 allows the use of Proposition 1.3.10 with normal subgroup B and D being the same as in the statement. Thus φ extends to an automorphism φ^* of $G \wr H$. Yet φ^* must be an inner automorphism of $G \wr H$ so we know then that φ is induced by $G \wr H$ and hence is an element of D. Therefore D is self-normalizing in Aut B and $\overline{D}$ is self-normalizing in Out B.

Suppose now that G is complete, that B is characteristic, and that the image of H is a self-normalizing subgroup of Out B. Again we let D be the image of $G \wr H$ in Aut B and $\overline{D}$ be the image of H in Out B. By Proposition 3.3.1 we know that D is self-normalizing in Aut B. Additionally, Proposition 3.3.5 holds so $\mathrm{Z}(G \wr H) = 1$ and we may again use Proposition 1.3.10 with $N = B$.

Let φ be an automorphism of $G \wr H$; we show φ is inner. Since the base is characteristic, the restriction of φ to B is defined. Proposition 1.3.10 yields the existence of an automorphism ψ of B that normalizes D and such that $\varphi_{|B} = \psi$. Yet D is self-normalizing, so there exists of an element bh in $G \wr H$ such that the action of ψ on B (as an automorphism) and the action of bh on B (via conjugation) are the same.

Since $\psi = \tau_{bh}$, we have $\varphi_{|B}\tau_{bh|B}^{-1} = (\varphi\tau_{bh}^{-1})_{|B}$ is the identity on B. Proposition 3.3.5(2) allows the use of Proposition 1.3.2, giving that $\varphi\tau_{bh}^{-1}$ is the identity on all of $G \wr H$. Thus φ is an inner automorphism and all automorphisms of $G \wr H$ are inner. Together with the fact that $\mathrm{Z}(G \wr H) = 1$, this yields that $G \wr H$ is complete. ∎

Theorem 3.3.7 is the motivation for our focus in Chapter 2 on answering the question of when the base of a permutational wreath product is complete. The original answers by P. Neumann and Y. Bodnarchuk pertained to cases with added restrictions on the action of H, whereas Theorem 3.3.7 makes no demands on the action of H other than it be faithful.

For that reason, we chose to investigate when an odd-dihedral group could be complete. These investigations led to the following two results.

Lemma 3.3.8. *If A is a finite abelian group of odd-order and the Sylow 2-subgroup of Aut A has order 2 then A is cyclic of prime power order.*

Proof. Since A is a finite abelian group of odd-order there exist distinct odd primes p_i as well as positive integers n and α_i such that $A = \prod_{i=1}^{n} C_{p_i{}^{\alpha_i}}$. In a direct product the automorphisms of any one component can extend to the entire direct product by defining the extension to be trivial on the other components. Thus $\prod_{i=1}^{n} \mathrm{Aut}\ C_{p_i{}^{\alpha_i}} \leq \mathrm{Aut}\ A$; therefore there are at least n factors of 2 dividing the order of Aut A. This tells us that $n = 1$; hence A is a cyclic group of prime power order. ∎

Proposition 3.3.9. *The only finite complete dihedral group is S_3.*

Proof. Suppose D is a complete dihedral group. Let A be the abelian normal subgroup of index 2 and $\langle x \rangle$ be a complement of A in D, i.e., x acts on the elements of A by inversion. If A has even order then A has an element of order two. This element would commute with x and hence $\mathrm{Z}(D)$ would be non-trivial, meaning D could not be complete.

Assume instead that A is of odd-order; in this case $C_D(A) \leq A$. The hypotheses of Proposition 3.1.17 then give the subgroup $\langle x \rangle$ is self-normalizing as a subgroup of Aut A. If the Sylow 2-subgroup of Aut A had order greater than 2 then there would be a subgroup of Aut A of order 4 containing x; this subgroup would centralize x so $\langle x \rangle$ could not be self-normalizing in Aut A. Thus the Sylow 2-subgroups of Aut A have order 2.

By the previous lemma A is cyclic of prime power order. Therefore Aut A is abelian and in order for $\langle x \rangle$ to be self-normalizing as a subgroup of Aut A, it must be the entirety of Aut A. Thus A must be a cyclic group of order 3, giving $D = S_3$. ∎

Proposition 3.3.9 yields the following corollary summarizing our result.

Corollary 3.3.10. *Let $G \wr H$ be the wreath product of a finite group G with a finite permutation group H, and let B represent the base of $G \wr H$. Whenever G is a complete group not equal to S_3 and H is isomorphic to a self-normalizing subgroup of the outer automorphism group of B then $G \wr H$ is complete.*

The restriction that $G \neq S_3$ is necessary. We will see an example where the base is not characteristic in Application 2 of the next subsection. However, we also noticed during our investigation that whenever we constructed an example of a complete wreath product $G \wr H$ with a directly indecomposable G then the outer automorphism group of the base would be a symmetric group. In fact, Theorem 3.1.6 from Section 3.1 helps prove that this is always the case when G is a directly indecomposable group.

Proposition 3.3.11. *Let G be a directly indecomposable centerless group and let n be a finite positive integer. Then the automorphism group of the direct product of n factors of G is $(Aut\ G) \wr S_n$. Moreover, if G is complete then the automorphism group is $G \wr S_n$ and the outer automorphism group is S_n.*

Proof. Let B be the direct product of n factors of G. Since G is centerless the homomorphism φ_{ij} defined in Theorem 3.1.6 (b) is always trivial for all values of i and j. Hence the direct decomposition of B is unique up to permutation, meaning that the only automorphisms of B which permute the coordinates arise from S_n. Therefore Aut $B \cong (\text{Aut } G) \wr S_n$.

If G is complete then Aut $G \cong G$ and so Aut $B \cong G \wr S_n$. As $\mathrm{Z}(B) = 1$ (refer to Theorem 3.3.5) we know that Inn $B \cong B$, which is the normal subgroup of $G \wr S_n$. This implies that Out $B \cong S_n$. ∎

We can then simplify Theorem 3.3.7 in the case of a directly indecomposable bottom group, noting that S_3 is still of concern since it is directly indecomposable.

Corollary 3.3.12. *Let G be a directly indecomposable finite group and let H be a finite permutation group. Then $G \wr H$ is complete if and only if G is complete but not equal to S_3 and the image of H in S_n is self-normalizing.*

3.3.3 Complete Standard Wreath Products

As in the last subsection, let G be a finite group. For this subsection let H be a finite group acting on itself by the right (or left) regular representation. We remind readers that in Chapter 2 we discussed how P. Neumann fully determined when the base of a standard wreath product was characteristic, and in the case of finite groups the trouble arose when G was odd-dihedral. In the last subsection we determined that the only complete dihedral group is S_3, and so we can say the following about complete standard wreath products.

Lemma 3.3.13. *Let G and H be finite groups with H acting on itself by the left (or right) regular representation. Let B be the base of $G \wr H$. Then $G \wr H$ is complete if and only if G is a complete group not equal to S_3 and the image of H in Out B is self-normalizing.*

In this subsection we further simplify the above result, proving that $G \wr H$ is complete if and only if G is a complete group not equal to S_3 and $H \cong C_2$. The proof of this fact was suggested by D. J. S. Robinson during his visit to Binghamton University in the fall of 2008. To refine our characterization we examine the self-normalizing subgroups of S_n, and show that such groups must in fact be isomorphic to C_2. The first step in the proof is an exercise found in standard references for permutation groups (see [9, Exercise 4.3.36], for example) and so we leave out the proof.

Exercise 3.3.14. The images of the left and right regular representations of a finite group commute. Moreover, their intersection is the image under either representation of the center of the group.

Lemma 3.3.15. *Suppose that H is a finite group whose images of the left and right regular representations are self-normalizing in S_H. Then the two images are equal as subgroups of S_H and H is abelian.*

Proof. Let H^ρ and H^λ be the images of the right and left regular representations of H, respectively. We assume these subgroups of S_H are self-normalizing. From Lemma 3.3.14 we know that

$$[H^\rho, H^\lambda] = 1.$$

The self-normalizing conditions yield that $H^\rho \leq H^\lambda$ and vice versa. These two subgroups are therefore equal and thus H is abelian by Lemma 3.3.14. ■

The ultimate goal is to show that a self-normalizing subgroup of S_n must be isomorphic to C_2 and while we have just shown that such a group must be abelian, we now consider the smaller case where it is known that H is of prime power order. In the course of the proof we will require the following fact, which is derived in the discussion present in [9, Section 6.2]

Lemma 3.3.16. *Let p be a prime. If P is a Sylow p-subgroup of S_k where $k = p$ or $k = p + 1$ then $|N_{S_k}(P)| = p(p - 1)$.*

Lemma 3.3.17. *Suppose that H is a group of prime power order for which the images of the left and right regular representations are self-normalizing in S_H. Then $H \cong C_2$.*

Proof. Let $|H|$ be p^k for a prime p and a positive integer k. Let H^ρ be the image of H under the right regular representation. By Lemma 3.3.14 we know that the images of the right and left regular representations coincide, so we focus our attention on H^ρ. Let P be a Sylow p-subgroup of S_H containing H^ρ. Then P is a k-fold wreath product of C_p with itself and there is a formula for the order of P (see [27, pages 10-11] for details).

$$|P| = p^{1+p+\cdots+p^{k-1}}$$

Since $N_{S_H}(H^\rho) = H^\rho$ we know that $N_P(H^\rho) = H^\rho$. Yet P is a p-group and H^ρ is a non-identity self-normalizing subgroup, thus $H^\rho = P$. Then

$$1 + p + \cdots + p^{k-1} = k,$$

which is only possible if $k = 1$. Therefore H is a cyclic group of order p and $S_H \cong S_p$.

Now H^ρ is a Sylow p-subgroup of S_p and by Lemma 3.3.16 we have that

$$|N_{S_H}(H^\rho)| = p(p - 1).$$

Since H^ρ is self-normalizing then $p(p - 1) = p$ and hence $p = 2$. ■

With these lemmas we can now prove the desired result, and as corollary we have the characterization of complete standard wreath products.

Theorem 3.3.18. *Suppose that H is a finite group for which the images of the left and right regular representations are self-normalizing in S_H. Then H must be isomorphic to C_2.*

Proof. By Lemma 3.3.14, the images of the left and right regular representations of H are equal as subgroups of S_H. For convenience we refer to this image simply as H^ρ and when necessary use the right action. Suppose that $|H|$ is not a power of a prime; then since H is abelian there exist $Q, R \trianglelefteq H$ such that $H = Q \times R$ and $|Q| = q^k$ for an odd prime q and k a positive integer.

Exercise 3.3.14 implies that the right and left regular representations of Q are equal, since Q is abelian. By Lemma 3.3.15, since $q \neq 2$ we know that this image is not self-normalizing. For convenience we will consider the image of the regular representation of Q in S_Q as Q^{ρ_0}.

Let π_0 be an element of $N_{S_Q}(Q^{\rho_0})$ that is not in Q^{ρ_0}. Define $\pi : H \to H$ by

$$(a\ b)\pi = (a)\pi_0\ b$$

where $a \in Q$ and $b \in R$. It's straightforward to show that π is a permutation of H and therefore an element of S_H. To show that π normalizes H^ρ, let $x \in Q$ and $y \in R$ so that $x\ y \in H$ and $(x\ y)^\rho \in H^\rho$. Consider the action of $\pi^{-1}(x\ y)^\rho\pi$ on an element of H, say $a\ b$. We use $\cdot$ to denote the group operation, differentiating between the group operation and the right regular action.

$$\begin{aligned}
(a\ b)\pi^{-1}(x\ y)^\rho\pi &= [(a)\pi_0^{-1}\ b](x\ y)^\rho\pi && \text{using the definition of } \pi \\
&= [(a)\pi_0^{-1}\ b \cdot x\ y]\pi && \text{by definition of } \rho \\
&= [(a)\pi_0^{-1}x\ by]\pi_0 && \text{since } H \text{ is a direct product} \\
&= [(a)\pi_0^{-1}x]\pi_0\ by && \text{by definition of } \pi \\
&= (a)[\pi_0^{-1}x^{\rho_0}\pi_0]\ by && \text{by definition of } \rho_0
\end{aligned}$$

Remember that π normalizes Q^{ρ_0}; there exists $w \in Q$ such that

$$w^{\rho_0} = \pi_0^{-1}x^{\rho_0}\pi_0.$$

Then

$$(a\ b)\pi^{-1}(x\ y)^\rho\pi = (a)w^{\rho_0}\ br = aw\ br = ab \cdot wr = (a\ b)(w\ r)^\rho.$$

Conjugating $(x\ y)^\rho$ by π gives another element of H^ρ, meaning that π is an element of $N_{S_H}(H^\rho) = H^\rho$.

Suppose that $\pi = (x\ y)^\rho$ where $x \in Q$ and $y \in R$. On one hand

$$(a\ b)\pi = (a)\pi_0\ b$$

and also

$$(a\ b)\pi = (a\ b) \cdot (x\ y) = ax\ by.$$

So we know that $y = 1$ and $ax = (a)\pi_0$ for all $a \in Q$. Therefore $\pi_0 = x^{\rho_0}$. This contradicts the choice of π_0, implying that H must be of prime power order. By Lemma 3.3.17 we know that H is isomorphic to C_2. ■

Theorem 3.3.19. *Let G be a finite group and let H act on itself via the right or left regular representation. Then $G \wr H$ is complete if and only if G is complete but not S_3 and $H = C_2$.*

We now may present two results following from this characterization of complete standard wreath products. The first is an independent proof that the Sylow 2-subgroups of S_{2^m} are self-normalizing. The second is an examination of the automorphism tower of $S_3 \times S_3$.

Application 1. Sylow 2-subgroups of S_{2^m} are self-normalizing.

In his 1925 paper [38], Louis Weisner proved using counting techniques and Sylow's theorems that the Sylow 2-subgroups of any symmetric group are self-normalizing. Here we present an independent proof in the case that the symmetric group is acting on 2^m points for $m > 0$.

The Sylow 2-subgroup of S_{2^m} is an m-fold wreath product of C_2 with itself, the proof of which is discussed in texts such as [27] and [22]. Let T_0 be the trivial group and let T_m be the Sylow 2-subgroup of S_{2^m} for all integers $m \geq 1$. For each value of m the group T_m acts transitively of degree 2^m.

Fix G as a complete group not equal to S_3. For each integer $i \geq 0$ define $H_i = G \wr T_i$. Our goal is to show that H_i is complete for each non-negative integer i, which we may do with the information we have developed about standard wreath products. Since H_i is not a standard wreath product we recursively define a family of standard wreath products G_i where i is a non-negative integer by setting $G_0 = G$ and $G_i = G_{i-1} \wr C_2$.

Recall that when a group X acts on a set Λ and a group Y acts on a set Ω, both as permutation groups, then $X \wr Y$ acts on $\Lambda \times \Omega$ by

$$(\lambda, \omega)by = (\lambda(\omega b), \omega y)$$

where by is an element of $X \wr Y$ and $(\lambda, \omega) \in \Lambda \times \Omega$. This actions is discussed in depth in [7] and [22]. Under this interpretation the wreath product operation is associative [22, Hilfsatz I.15.4]. It is easy to see then that $H_i \cong G_i$ for all non-negative integers i.

Furthermore since $G \neq S_3$ we may apply Theorem 3.3.7 to each group in the collection $\{G_i\}$, yielding a family of complete groups. Hence H_i is also complete for each value of i. Our desired result then follows.

Proposition 3.3.20. *The Sylow 2-subgroups of S_{2^m} are self-normalizing for all positive integers m.*

Proof. The group H_m is complete, but is not a standard wreath product in this definition. The group T_m is acting on a set of size 2^m and not on itself, which would be a set of size $2^{2^{m-1}+\cdots+2+1}$. Theorem 3.3.7 then states that the top group is self-normalizing as a subgroup of S_{2^m} and the proposition is proven. ∎

Application 2. The Automorphism Tower of $S_3 \times S_3$

Our initial examination was of the automorphism tower of $S_3 \wr C_2$. Theorem 3.3.7 implies that $S_3 \wr C_2$ is not complete since the base is not characteristic. However Proposition 3.3.11 shows that $S_3 \wr C_2$ is the automorphism group of the centerless group $S_3 \times S_3$. Hence we investigated the automorphism tower of $S_3 \wr C_2$ unaware that we were also investigating the automorphism tower of $S_3 \times S_3$.

We show that the automorphism tower of $S_3 \times S_3$ has height equal to 2, which means that Aut (Aut ($S_3 \times S_3$)) is complete. In other words, Aut ($S_3 \wr C_2$) is complete. The process uses the same approach as for complete wreath products and $[C_3 \times C_3]\mathrm{GL}_2(3)$. For ease of notation let $W = S_3 \wr C_2$ and $X = C_3 \times C_3$.

Lemma 3.3.21. *The subgroup X is characteristic in W and the centralizer of X in W is X.*

Proof. First, since X has order 3^2 and W has order $2^3 \cdot 3^2$, the subgroup X is a normal Sylow 3-subgroup of W and hence is characteristic.

It is clear that the centralizer of X in $S_3 \times S_3$ is X. Moreover the top group C_2 does not centralize the non-diagonal elements of X. Thus only elements of X can centralize X. ∎

Corollary 3.3.22. *The center of W is trivial.*

By Proposition 1.3.1, as W is centerless we may embed W injectively in its automorphism group by identifying an element w with the inner automorphism τ_w it induces.

Let $w \in W$ and $\varphi \in \text{Aut } W$. Let $\widehat{X}$ be the image of X under this embedding, so that $\widehat{X} \leq \text{Inn } W$. Let $w \in W$ and $\varphi \in \text{Aut } W$. Under the embedding the action of φ on w as an automorphism becomes the action of φ on w via conjugation. This yields $\varphi(w) = w^{\varphi} = (\tau_w)^{\varphi}$.

The approach we use identifies a critical characteristic subgroup of Aut W, which will be $\widehat{X}$. In order for the approach to work, we need to show that $\widehat{X}$ either has a trivial centralizer in W or is equal to its centralizer. Since X, and hence $\widehat{X}$, is abelian, we prove the latter possibility is true.

Lemma 3.3.23. *The centralizer in Aut W of $\widehat{X}$ equals $\widehat{X}$.*

Proof. The subgroup X is abelian, therefore $\widehat{X}$ is abelian and $\widehat{X} \leq C_{\text{Aut } W}(\widehat{X})$. Let φ be an element of this centralizer. Then φ is an automorphism of W fixing each element of X. As W/X has order 8, by Lemma 1.3.7 we have that $H^1(W/X, X) = 1$. With this and Lemma 3.3.21 the hypotheses of Proposition 1.3.6 are satisfied. Therefore φ must be an inner automorphism of W. Using the embedding, φ is then an element of Inn W that centralizes $\widehat{X}$. Since X is self-centralizing in W, this yields φ is actually an element of $\widehat{X}$. ■

Much of the structure of W, including order, was established in Proposition 2.3.1 from Chapter 2. As a result the following lemma is easy.

Lemma 3.3.24. *The order of Aut W is 144, and $\widehat{X}$ is characteristic in Aut W.*

Proof. Proposition 2.3.1 from Chapter 2 states that

$$|\text{Aut } W| = 8|\text{Aut } C_3||C_3|^2 = 8 \cdot 2 \cdot 3^2 = 144.$$

Therefore $\widehat{X}$ is a Sylow 3-subgroup of Aut W. Yet X is characteristic in W, so we have that $\widehat{X} \trianglelefteq \text{Aut } W$ and thus $\widehat{X}$ is characteristic. ■

Since $\widehat{X} \cong X$ the automorphism group of $\widehat{X}$ is isomorphic to Aut X, which is $\text{GL}_2(3)$. Furthermore, Proposition 1.3.1 describes how Aut W acts as automorphisms of $\widehat{X}$ via conjugation, with kernel equal to $C_{\text{Aut } W}(\widehat{X}) = \widehat{X}$.

Lemma 3.3.24 tells us that Aut $W/\widehat{X}$ has order 16, hence it is isomorphic to a Sylow 2-subgroup of $\text{GL}_2(3)$. These subgroups are maximal (of index 3) and are non-normal;

they are therefore self-normalizing. This allows us to see that the automorphisms of $\widehat{X}$ which normalize the quotient Aut $W/\widehat{X}$ are precisely those in the image of Aut $W/\widehat{X}$.

We use that information, as well as Proposition 3.1.9, to prove the final result of this section. Recall that this proposition of Wielandt states that if $\mathrm{Z}(G) = 1$ then $C_{\text{Aut } G}(\text{Inn } G) = 1$, hence $\mathrm{Z}(\text{Aut } G) = 1$.

Theorem 3.3.25. *The group Aut W is complete.*

Proof. The center of W is trivial, so Proposition 3.1.9 yields that the center of Aut W is trivial as well. As $\widehat{X}$ is a normal Sylow 3-subgroup, by Lemma 1.3.7 the first cohomology group $H^1(\text{Aut } W/\widehat{X}, \widehat{X})$ is trivial. As we proved in Lemma 3.3.23 that $\widehat{X}$ is equal to its own centralizer in Aut W, the situation exactly satisfies the hypotheses of Proposition 1.3.6.

Let φ be an automorphism of Aut W and consider $\varphi_{|\widehat{X}}$. This restriction is an automorphism of $\widehat{X}$ and hence acts like a member of Aut $\widehat{X} \cong \mathrm{GL}_2(3)$. It must also normalize Aut $W/\widehat{X}$, which is isomorphic to a self-normalizing Sylow 2-subgroup of $\mathrm{GL}_2(3)$. Thus there exists $z \in \text{Aut } W$ for which $\varphi_{|\widehat{X}} = (\tau_z)_{|\widehat{X}}$.

Then $\varphi\tau_z^{-1}$ fixes $\widehat{X}$ element-wise. By Proposition 1.3.6 the automorphism $\varphi\tau_z^{-1}$ is inner on Aut W and therefore φ is an inner automorphism of Aut W. Since φ was an arbitrary automorphism, all automorphisms of Aut W are inner. Thus Aut W is complete. ∎

We therefore see that the automorphism tower of $S_3 \times S_3$ has height 2, ending in Aut $(S_3 \wr C_2)$.

3.3.4 Complete Frobenius Groups

While the previous parts of this section were focused on wreath products, we turn now to Frobenius groups and characterization of complete Frobenius groups.

Definition 3.3.26. A group G is a *Frobenius group* if there exists a non-trivial proper normal subgroup K such that for every non-identity element $k \in K$, the centralizer of k in G is strictly contained in K. If G is a Frobenius group then K is called the *Frobenius kernel of* G and any complement of K is called a *Frobenius complement.*

There are many examples of Frobenius groups, including the odd-dihedral groups we examined earlier. Another familiar example is the holomorph of a cyclic group of prime order. Of course, this latter group was proved to be complete by Burnside (Proposition 3.1.5). A more complex example of a Frobenius group is the group with underlying set $\{x \mapsto ax + b \mid a, b \in \mathbb{F}_p, a \neq 0, p \text{ is a prime}\}$ under composition.

Frobenius groups played a central role in M. Suzuki's 1957 paper [34] ruling out the existence of odd-order simple CA-groups (groups wherein every centralizer is abelian). Suzuki's result in turn is a cornerstone for Feit and Thompson's result that all non-abelian simple groups are of even order [10]. With this history it is no surprise that the literature on Frobenius groups is quite rich, and contains or implies most of the results we summarize in the next theorem. For additional facts about Frobenius groups we refer readers to [27] or [13].

For the rest of this subsection let G be a Frobenius group with Frobenius kernel K and complement E.

Theorem 3.3.27. *Let G be a Frobenius group with Frobenius kernel K and complement E. The following facts are true.*

(1) *The centralizer in G of K is contained in K; in fact it is the center of K.*

(2) *The center of G is trivial.*

(3) *The orders of E and K are relatively prime.*

(4) *The Frobenius kernel is characteristic in G.*

(5) *The cohomology group $H^n(E, Z(K)) = 1$ for all $n \geq 1$.*

(6) *The quotient $G/Z(K)$ is isomorphic to a subgroup of Aut K and E is isomorphic to a subgroup of Out K.*

The first part of Theorem 3.3.27 follows directly from the definition of Frobenius group, and part (2) falls quickly in line. Part (3) is immediately obtained from the fact that the center of a non-trivial p-group is non-trivial, which will imply that K is a Hall subgroup of G. Parts (4) and (5) are direct consequences of part (3). The last part of Theorem 3.3.27 is clear from Proposition 1.3.1, and from the fact that

$$(G/\mathrm{Z}(K))/(K/\mathrm{Z}(K)) \cong G/K \cong E.$$

We refer to the image of $G/\mathrm{Z}(K)$ in Aut K as D and the image of E in Out K as $D/\mathrm{Inn}\ K$.

Lemma 3.3.28. *If G is complete then D is a self-normalizing subgroup of Aut K.*

Proof. Assume G is complete. Since Aut $G =$ Inn G, the subgroup of automorphisms of K induced by Aut G is exactly D. We need to show that $N_{\text{Aut }K}(D) = D$.

Theorem 3.3.27 tells us that the situation satisfies the hypotheses of Proposition 1.3.10 with $N = K$ and D in the statement of the proposition being the same as our D. The converse of Proposition 1.3.10 states that if $\alpha \in N_{\text{Aut }K}(D)$ then there exists an extension of α to an automorphism of G that normalizes K. In other words, every element of $N_{\text{Aut }K}(D)$ induces an automorphism of G.

Suppose that $\alpha \in N_{\text{Aut }K}(D)$. Since G is complete there exists $g \in G$ such that the extension of α to G is equal to τ_g. Thus the extension α^* acts on K via conjugation, meaning that its restriction is one of the automorphisms of K induced by G. Thus the restriction α is an element of D. We have shown that $N_{\text{Aut }K}(D) \leq D$. Therefore the two subgroups are equal and the lemma is established. ■

Theorem 3.3.29. *The group G is complete if and only if the image of E in Out K is self-normalizing.*

Proof. Suppose that G is complete. Lemma 3.3.28 implies that $N_{\text{Aut }K}(D) = D$. Proposition 3.3.1 then gives, with $N =$ Inn K and U the image of E in Aut K, that $\overline{D}$ is self-normalizing in Out K. Thus the proof of one implication is finished.

Now suppose that $\overline{D}$ is self-normalizing in Out K. Then by Proposition 3.3.1 we have that D is self-normalizing in Aut K. The center of G is trivial; thus we need only consider φ, an automorphism of G, with the goal of showing that φ is inner.

Since K is characteristic in G, restrict φ to K. As $\varphi_{|K}$ must normalize the image of $G/\mathrm{Z}(K)$ in Aut K we have that $\varphi_{|K}$ is in $N_{\text{Aut }K}(D) = D$. Thus there exists an element $g \in G$ for which $\tau_g = \varphi_{|K}$.

This in turn implies that $\varphi\tau_{g^{-1}}$ is trivial on K. Parts (4) and (5) of Theorem 3.3.27 exactly satisfy the requirements of Proposition 1.3.6, which gives that $\varphi\tau_{g^{-1}}$ is inner on G. Then φ is an inner automorphism of G, as well. Therefore all automorphisms of G are inner and G is complete. ■

Theorem 3.3.29 implies that the holomorph of a nilpotent group is never a Frobenius group, otherwise it would be complete and this would contradict Mills' result that the holomorph of a non-abelian group is never complete [25, p. 428-429]. Of course, this fact is well-known due to the exercise:

Exercise 3.3.30. If G is any non-trivial finite group such that every non-trivial automorphism of G is fixed point free, then $G \cong Z_p$ for some prime p.

The value of Theorem 3.3.29 is subtle. For example, for any prime p the automorphism group of C_p is abelian. Therefore Aut C_p does not have proper self-normalizing subgroups and hence we have the next corollary.

Corollary 3.3.31. *Let p be an odd prime and m divide $p-1$, so that C_m is isomorphic to a subgroup of Aut C_p and acts on C_p as automorphisms. The group $C_p \rtimes C_m$ is complete if and only if $m = p-1$.*

More generally, if a Frobenius kernel K has an abelian automorphism group then only the holomorph of K could be possibly complete – and it is unlikely that the holomorph is even a Frobenius group.

Notice the similarity between the characterization for complete Frobenius groups and that for complete wreath products. Both of these groups are semidirect products with the normal subgroup either the Frobenius kernel or the base of the wreath product. The requirement that the base of a wreath product be characteristic is not present in the case for Frobenius groups since the Frobenius kernel is always characteristic. Yet in both characterizations we see the requirement that the image of complement in the outer automorphism group of the normal subgroup be self-normalizing, following from Propositions 3.3.1 and 1.3.10 which we saw at the start of the section.

Conclusions

"In mathematics the art of proposing a question must be held of higher value than solving it." – Georg Cantor (1845-1918)

We finish by posing some of the questions our work has raised. With regards to wreath products, we have ascertained the structure of the bottom group when the base of a finite permutational wreath product is not characteristic. We've also made progress in understanding the automorphism group of a wreath product with a non-characteristic base. Our methods depend on techniques of finite group theory; it is probably still true that if the base of an infinite permutational wreath product $G \wr H$ is not characteristic then G is *special dihedral*. This fact has yet to be established, though.

Our methods may or may not extend to the generalizations of wreath products. Constructions like twisted wreath products are not as well understood as permutational wreath products; our method of examining the centralizers of elements may be helpful in further examining such generalizations.

With regards to complete groups, there is still much to be uncovered. While our approach to complete groups is too specific to extend to all semidirect products, we see a trend in complete semidirect products of examining a crucial normal subgroup. In our case, this normal subgroup was the base of a wreath product or the Frobenius kernel. Notice that in a Frobenius group the Frobenius kernel is always characteristic, but in a wreath product our crucial normal subgroup is the base and we must require it to be characteristic. In fact, requiring that the group G in a wreath product be complete achieves two properties also inherent to Frobenius kernels: (1) the centralizer of the base in $G \wr H$ will be contained in the base and (2) the associated first and second cohomology groups will be trivial. This idea may extend to other semidirect products if the normal subgroup is characteristic.

There is also the question of how many examples our classifications can produce. For Frobenius groups, much is known about the possibilities for a Frobenius kernel. This leads to the question, how much is known about the automorphism group of a Frobenius kernel? Are there known non-trivial self-normalizing subgroups of such groups?

To build a complete wreath product, find a directly indecomposable complete group $G \neq S_3$ to be the bottom group and a self-normalizing subgroup of a symmetric group to be the top group. To build a complete *solvable* group in this manner, find a solvable complete group and a solvable self-normalizing subgroup in S_n. While Weisner [38] proved that the Sylow 2-subgroups of S_n are always self-normalizing, what other subgroups of S_n are self-normalizing? Is it true that the the maximal solvable subgroups of S_n are self-normalizing? Or, is there a way to tell which maximal solvable subgroups of S_n are self-normalizing?

Our results in Section 3.2 rely on a complete odd-order group having two non-isomorphic minimal normal subgroups. Yet the known examples of complete odd-order groups have a unique minimal normal subgroup. Do odd-order complete groups, other than direct products, with more than one non-isomorphic minimal normal subgroup exist?

Another question our work raises is, what do the normal subgroups of a complete group look like? More generally, what are the subnormal subgroups of a complete group? These questions are most likely to be answered in the case of a complete wreath product since so much is known about the normal subgroups of wreath products.

One may also ask if our work on complete groups can be extended to infinite groups. In fact, one may ask well-understood examples of infinite complete groups exist. For example, it is known that the automorphism group of a free group is complete but well-known is the structure of the automorphism group of a free group? Similarly, we know that the automorphism group of a non-abelian simple group is complete but what is known about these groups? Can we link the structure of infinite complete groups to the structure of finite complete groups?

These questions, and more, have yet to be answered in the study of automorphisms of groups.

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