Chapter 10

Exponential and Logarithmic Functions

In this chapter we look at two new kinds of functions, the exponential function and the logarithmic function. We begin with the exponential function.

10.1 Exponential Functions

10.1.1 Definition

In an exponential function, the independent variable is in the exponent.

Definition 10.1.1. *An exponential function is a function in the form $f(x) = a^x$ where a is a constant real number and $a > 0$ and $a \neq 1$. The number a is called the base of the function.*

The domain for this function is $\Re$, the set of all real numbers.

So, $f(x) = 2^x$ and $g(x) = (\frac{3}{7})^x$ and $h(x) = (\sqrt{5})^x$ are all examples of exponential functions.

Why do you suppose that we do not allow a to have the value 1? Consider what some of the values would be for $f(x) = 1^x$: $f(2) = 1^2 = 1$, and $f(\frac{1}{2}) = 1^{\frac{1}{2}} = 1$, and $f(0) = 1^0 = 1$. Indeed, for any value of x, $f(x) = 1$. This then is a linear function, the function $f(x) = 1$, so we do not include it in the class of exponential functions.

Why do you suppose that we do not allow a to be a negative number? One obvious reason is that we would have trouble with some x values less than 1. If $a = -4$ and $x = \frac{1}{2}$ we would not be able to evaluate $a^x = (-4)^{\frac{1}{2}} = \sqrt{-4}$. Certainly $\sqrt{-4}$ is undefined in $\Re$. So, we keep our life simple and only allow a to be a positive number, but not equal to 1.

10.1.2 Characteristics and Graphs

A Close Look at $f(x) = 2^x$

Let's look in detail at the particular exponential function $f(x) = 2^x$. We'll start by looking at some specific values of the function:

- Some positive integer values for x: $f(1) = 2^1 = 2$; $f(2) = 2^2 = 4$; $f(3) = 2^3 = 8$; $f(4) = 16$; $f(10) = 1,024$; $f(20) = 1,048,576$; $f(50)$ is a really big number with 16 digits. For $x > 10$ the use of a calculator is forgivable.
- We certainly don't want to forget our y-intercept, where $x = 0$: $f(0) = 2^0 = 1$.
- And now some negative integer values for x: $f(-1) = 2^{-1} = \frac{1}{2}$; $f(-2) = 2^{-2} = \frac{1}{4}$. Indeed we recall that $a^{-p} = \frac{1}{a^p}$ so we can use our work from above to easily get: $f(-3) = \frac{1}{f(3)} = \frac{1}{8}$; $f(-4) = \frac{1}{16}$; $f(-10) = \frac{1}{1,024}$; $f(-20) = \frac{1}{1,048,576}$; and $f(-50)$ is a fraction with a 1 in the numerator and a really big number with 16 digits in the denominator.
- Now let's look at some non-integer values of x: $f(\frac{1}{2}) = 2^{\frac{1}{2}} = \sqrt{2} \approx 1.414$. Similarly, $f(\frac{1}{3}) = 2^{\frac{1}{3}} = \sqrt[3]{2} \approx 1.260$; $f(\frac{1}{4}) = \sqrt[4]{2} \approx 1.189$; $f(\frac{1}{10}) = \sqrt[10]{2} \approx 1.072$. We use a calculator here to get decimal approximations because we will use them in our study below.
- We use the same "reciprocal trick" (and a calculator) to get function values for x's that are negative non-integer values: $f(-\frac{1}{2}) = \frac{1}{\sqrt{2}} \approx 0.707$; $f(-\frac{1}{3}) = \frac{1}{\sqrt[3]{2}} \approx 0.794$; $f(-\frac{1}{4}) \approx 0.841$; $f(-\frac{1}{10}) \approx 0.933$.

We now summarize in a list all of the data we have determined so far.

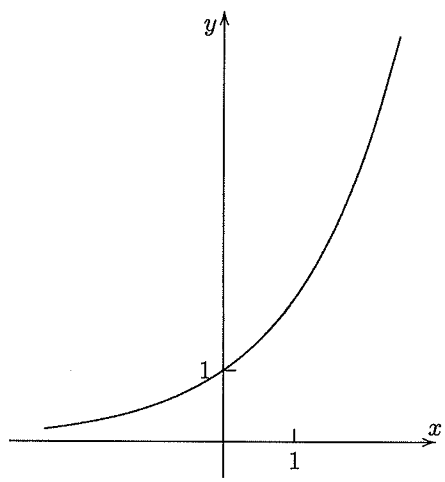
x	$f(x) = 2^x$	x	$f(x) = 2^x$
-20	$\frac{1}{1,048,576}$	$\frac{1}{10}$	1.072
-10	$\frac{1}{1,024}$	$\frac{1}{4}$	1.189
-4	$\frac{1}{16}$	$\frac{1}{3}$	1.260
-3	$\frac{1}{8}$	$\frac{1}{2}$	1.414
-2	$\frac{1}{4}$	1	2
-1	$\frac{1}{2}$	2	4
$-\frac{1}{2}$	0.707	3	8
$-\frac{1}{3}$	0.794	4	16
$-\frac{1}{4}$	0.841	10	1,024
$-\frac{1}{10}$	0.933	20	1,048,576
0	1		

Look carefully at this list of function values. We will make some observations. You should decide if you think that the observation is just peculiar to the specific data chosen, or if it will hold for the whole function $f(x) = 2^x$ or if this observation might hold for all exponential functions.

- First we observe that all of the $f(x)$ values are positive. Is there a value of x that would make 2^x negative? or make 2^x zero? Can you think of a value for a and a value for x so that a^x is negative? or zero? Remember that $a > 0$ and $a \neq 1$.
- Next we observe that the $f(x)$ values are increasing as x increases. When x is the smallest number in the list, $f(x)$ is the smallest number. When x is the largest number, $f(x)$ is the largest number. Do you think this pattern holds for x values not on the list? Is it reasonable that the value of 2^x should increase if x increases? Do you think this holds for a^x?
- We see that $f(0) = 1$. Is this true for all values of a? Is it true that $a^0 = 1$ for all values of a where $a > 0$ and $a \neq 1$?

- Finally, look at the pattern of values as x gets very large or x gets very small. What do you suppose happens to $f(x)$? We have already seen that $f(x) = 2^x$ is increasing, but is it doing so without bound or can we expect its graph to have a horizontal asymptote as $x \to \infty$? What about as $x \to -\infty$? Can you make a general statement about a^x?

We can use (some of) the data above to draw a sketch of the graph of $f(x) = 2^x$ (below). It has been stated earlier that graphing a function is more than just plotting a few points and then "playing dot-to-dot." However, it does turn out this time that the graph for this function is indeed a smooth curve that behaves nicely (it is all connected; doesn't jump around or have "holes" in it).



$y = 2^x$

Irrational Exponents

We have said that the domain for $f(x) = 2^x$ is $\Re$. Yet all of the data points that we plotted were rational x values. We had a variety of values: positive, negative, integer, non-integer. From our study of exponents in Chapter 1 we know how to deal with these. But, how do we deal with x values that are irrational, such as $x = \sqrt{3}$ or $x = \pi$? What does 2^π mean? It does not mean 2 multiplied times itself π times. It does not mean the "πth" root of 2. We can use the graph of $f(x) = 2^x$ to get a value for 2^π. 2^π is simply the y-value on the graph when the x value is π.

You might think that this is a "cheating" way to answer the question, "What do we mean by 2^π?" But, at least it is a reasonable answer. Since the graph of $f(x) = 2^x$ is smooth and increasing, and since $3.1 < \pi < 3.2$, we would want the value of 2^π to be somewhere between the values of $2^{3.1}$ and $2^{3.2}$. We know how to interpret $2^{3.1}$ and $2^{3.2}$ because these exponents are rational. Since $3.1 = \frac{31}{10}$ and $3.2 = \frac{32}{10}$ we understand $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$ and $2^{3.2} = 2^{\frac{32}{10}} = \sqrt[10]{2^{32}} \approx 9.198$. Our smooth, connected graph tells us that there IS a value for 2^π and we have figured that it must be between 8.574 and 9.198. We can get closer to the actual value of 2^π by simply choosing rational numbers closer to π than are 3.1 and 3.2. With calculus we have a way of squeezing the interval so closely around π that we say we can know the actual value of 2^π.

So, what is the point of all of this? Certainly from a practical standpoint we will simply use a calculator to get $2^\pi \approx 8.825$. But what we have here is a way to interpret an irrational exponent. The smooth connectedness of the graph of $f(x) = a^x$ gives us a way to understand the values of numbers in the form a^x where the exponent is irrational. We can consider all irrational exponents in the same way that we did here with π.

Exponential Functions with base $a > 1$

Now we will look at some exponential functions besides just $f(x) = 2^x$. Complete the following table and use it to sketch the graphs of $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = 4^x$ on the same set of axes.

x	$f(x) = 2^x$	$g(x) = 3^x$	$h(x) = 4^x$
-3	$\frac{1}{8}$		
-2	$\frac{1}{4}$		
-1	$\frac{1}{2}$		
$-\frac{1}{2}$	0.707		
0	1		
$\frac{1}{2}$	1.414		
1	2		
2	4		
3	8		

Compare the three graphs:

- When $x < 0$, which function has the highest function values? which has the lowest?
- When $x = 0$, what do you notice?
- When $x > 0$, which function has the highest function values? which has the lowest?

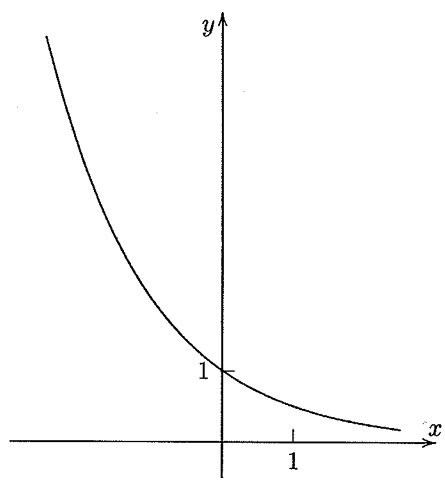
Make a general statement to describe your findings.

Use this statement to sketch in the graph of $k(x) = (\frac{7}{2})^x$ without plotting points.

Exponential Functions with base $0 < a < 1$

You will recall that the definition of exponential function allows for the base a to be any positive number except 1. So far we have dealt only with a values greater than 1. Let's consider the exponential function $f(x) = (\frac{1}{2})^x$. If we make a table of values for this function and look at its graph we see that it is a little different from the ones studied previously, although some of the numbers look quite familiar.

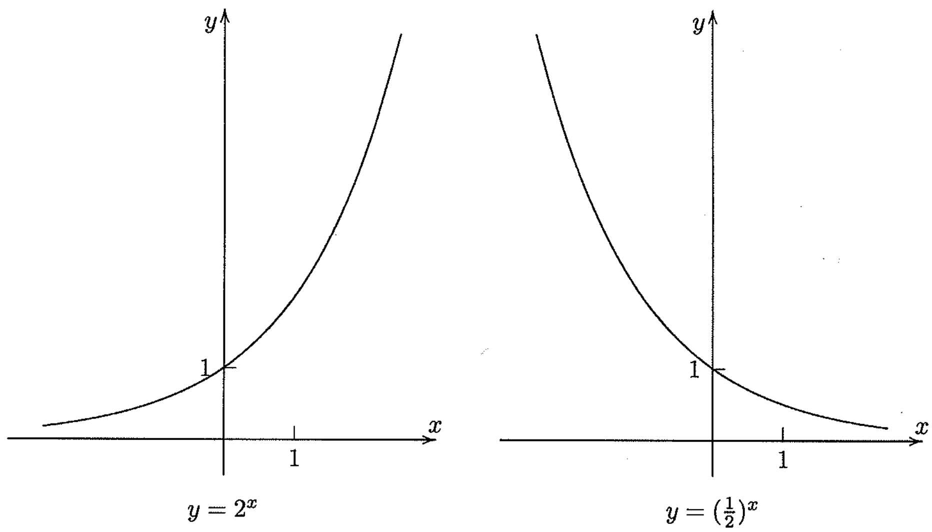
x	$f(x) = (\frac{1}{2})^x$	x	$f(x) = (\frac{1}{2})^x$
-20	$1,048,576$	$\frac{1}{10}$	0.933
-10	$1,024$	$\frac{1}{4}$	0.841
-4	16	$\frac{1}{3}$	0.794
-3	8	$\frac{1}{2}$	0.707
-2	4	1	$\frac{1}{2}$
-1	2	2	$\frac{1}{4}$
$-\frac{1}{2}$	1.414	3	$\frac{1}{8}$
$-\frac{1}{3}$	1.260	4	$\frac{1}{16}$
$-\frac{1}{4}$	1.189	10	$\frac{1}{1,024}$
$-\frac{1}{10}$	1.072	20	$\frac{1}{1,048,576}$
0	1		



$y = (\frac{1}{2})^x$

The values for the exponential function with base $a = \frac{1}{2}$ and the values for the exponential function with base $a = 2$ are simply reciprocals of each other. We can explain this algebraically: $(\frac{1}{2})^x = \frac{1^x}{2^x} = \frac{1}{2^x}$.

We can take this a step further: $(\frac{1}{2})^x = \frac{1}{2^x} = 2^{-x}$. In Chapter 3 we learned that for any function f, the graph of $f(x)$ and the graph of $f(-x)$ are reflections of each other about the y-axis. Look at the graphs below for $f(x) = 2^x$ and $f(-x) = 2^{-x} = (\frac{1}{2})^x$.



What do you suppose the graph of $f(x) = (\frac{1}{3})^x$ looks like? How will it compare to exponential graphs with bases of 3 or $\frac{1}{2}$?. Test your thoughts by sketching (you can do it now without plotting points) the graphs for 2^x and 3^x and then sketching in the graphs for $(\frac{1}{2})^x$ and $(\frac{1}{3})^x$ by using the reflections. Be careful. The only place that any graph will cross another is at the point $(0, 1)$.

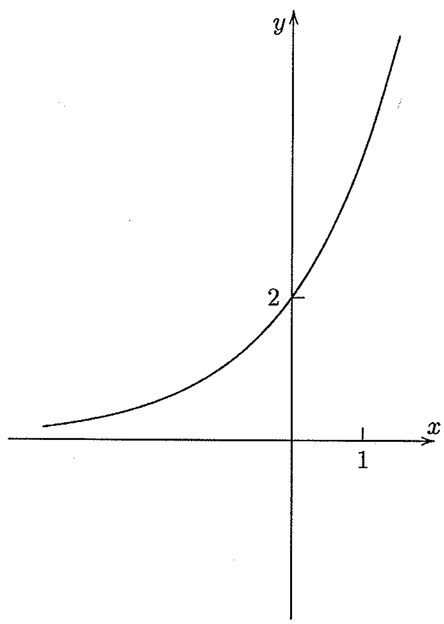
Since we now have some exponential functions whose graphs decrease, you might want to revisit the conclusions you drew concerning exponential functions. We can now state some definite facts about exponential functions.

Important Idea 10.1.1.
Functions in the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$:

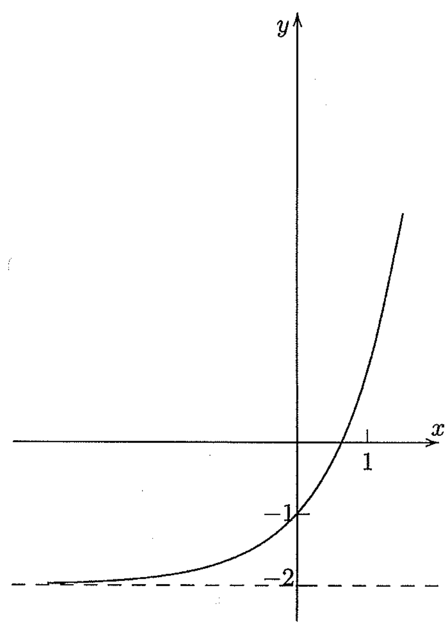
1. *have domain $\Re$,*
2. *have range $(0, \infty)$ (the function values are always positive),*
3. *have y-intercept $(0, 1)$,*
4. *and, when $a > 1$, we have:*
   (a) *f is increasing,*
   (b) *as $x \to \infty$, $f \to \infty$, and*
   (c) *as $x \to -\infty$, $f \to 0$ (there is a left horizontal asymptote $y = 0$),*
5. *and, when $0 < a < 1$ we have:*
   (a) *f is decreasing,*
   (b) *as $x \to -\infty$, $f \to \infty$, and*
   (c) *as $x \to \infty$, $f \to 0$ (there is a right horizontal asymptote $y = 0$).*

Variations

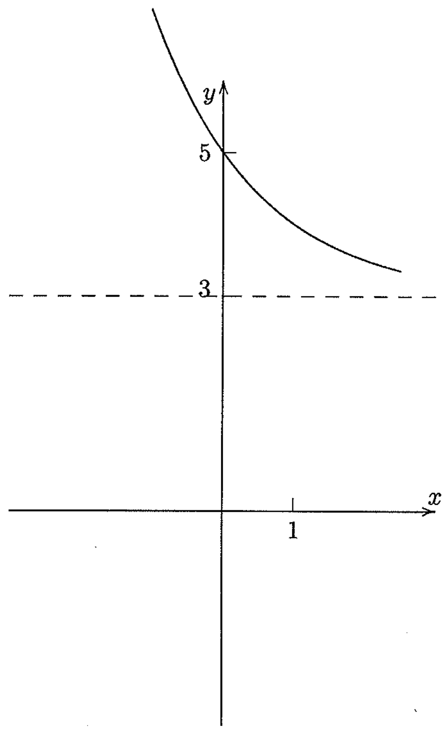
You should become familiar with the graphs of the exponential functions. Adopt them into the "family of functions" studied in chapter 4. These exponential functions can be moved and stretched by applying the same transformations that were used to operate on other functions. Below are some examples.



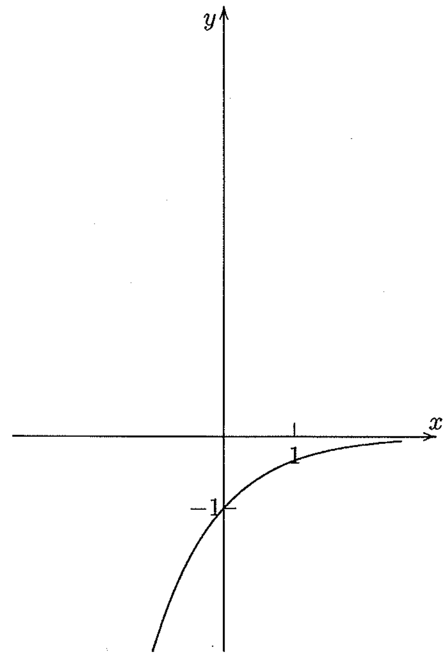
$y = 2^{x+1}$



$y = 3^x - 2$



$y = (\frac{1}{2})^{(x-1)} + 3$



$y = -(\frac{1}{3})^x$

10.1.3 The number 'e'

You are familiar with the number π. While you might not know how the actual value was arrived at you do know that it is the number that gives the ratio of the length of the arc of a semi-circle to the radius of the circle. You know that while π is irrational it can be approximated to any desired level of accuracy. In this course we are usually content with $\pi \approx 3.14$.

We now introduce another helpful irrational number. This number is called 'e.' The geometric development of e is beyond the scope of this course[1] but you will enjoy seeing its development in calculus. This number is mentioned in this course so that when you encounter it in calculus you will be comfortable with it, as you now are with π. The number e has its applications mainly with exponential and logarithmic functions.

The number $e \approx 2.71828$. Just as you think of π as being a little more than 3, you can think of e as being a little less than 3. So, now when you are asked to "pick a number between 1 and 10" you have another alternative: e. While e is a truly wonderful number, it is still a number, bound by all the same algebraic rules as any other constant, so don't let its appearance bother you.

Since e is a real number and $e > 0$ and $e \neq 1$ we can use it as the base for our exponential function.

You know what the graphs of $f(x) = 2^x$ and $f(x) = 3^x$ look like. On the blank page opposite, draw these graphs on the same set of axes. You know that $2 < e < 3$. Use this information to sketch

[1] Author's irrelevant note: I think I've read the phrase "beyond the scope of this course" in every math book I've ever read. I've always wanted to be able to write that myself.

the graph of $f(x) = e^x$.

Comprehension Check 10.1.

1. *Sketch the graph of* $f(x) = e^x$ *on a new set of axes.*
2. *Sketch the graph of* $f(x) = e^{-x}$.
3. *Sketch the graph of* $f(x) = e^{-x} + 1$.

10.1.4 Solving Simple Exponential Equations

We can see that the graphs of the exponential functions pass the Horizontal Line Test. This means that the functions must be one-to-one. From this we immediately get the following result:

Important Idea 10.1.2.

If $a^x = a^y$, *then* $x = y$. *AND* *If* $x = y$, *then* $a^x = a^y$.

We can use this idea to solve equations that involve simple exponential functions. In each example below we want to solve for x.

Example 10.1.1.

$$\begin{aligned} 10^x &= 1000 \\ 10^x &= 10^3 \\ x &= 3 \end{aligned}$$

Example 10.1.2.

$$\begin{aligned} 2^x &= \frac{1}{4} \\ 2^x &= 2^{-2} \\ x &= -2 \end{aligned}$$

Example 10.1.3.

$$\begin{aligned} 2^{3x+1} &= \sqrt{2} \\ 2^{3x+1} &= 2^{\frac{1}{2}} \\ 3x+1 &= \frac{1}{2} \\ 3x &= -\frac{1}{2} \\ x &= -\frac{1}{6} \end{aligned}$$

Sometimes we will need to rewrite the exponential term so that it can be used with previously learned equation solving methods. The next two examples use The Property of Zero; the second one is a quadratic "in disguise."

Example 10.1.4.

$$x^2(5^{x+2}) - 9(5^x) = 0$$
$$x^2(5^2 5^x) - 9(5^x) = 0$$

$$25x^2(5^x) - 9(5^x) = 0$$
$$(5^x)(25x^2 - 9) = 0$$
$$(5^x)(5x + 3)(5x - 3) = 0$$

So, we get: $(5^x) = 0$ *or* $(5x + 3) = 0$ *or* $(5x - 3) = 0$.
Since there are no values for x *where* $(5^x) = 0$, *our final solutions are* $x = -\frac{3}{5}$ *or* $x = \frac{3}{5}$.

Example 10.1.5.

$$4^x + 2^x - 2 = 0$$
$$(2^2)^x + 2^x - 2 = 0$$
$$(2^x)^2 + (2^x) - 2 = 0$$

If we let $u = 2^x$ *we get:*

$$u^2 + u - 2 = 0$$
$$(u + 2)(u - 1) = 0$$

So, $u = -2$ *or* $u = 1$. *This gives:* $2^x = -2$ *or* $2^x = 1$.
There are no solutions for $2^x = -2$, *so our only solution is* $x = 0$.

10.2 Logarithmic Functions

10.2.1 Definition

We notice that the graph of the exponential function $f(x) = a^x$ passes the Horizontal Line Test. Therefore, it is a one-to-one function and so has an inverse function, $f^{-1}(x)$.

The definition of inverse function tells us that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. For $f(x) = a^x$ this means $a^{f^{-1}(x)} = x$ and $f^{-1}(a^x) = x$. This inverse funtion for the exponential function has a special name. It is called the *logarithm* function. We write it as $f^{-1}(x) = \log_a(x)$ and read it "log, base a, of x." Using this notation, $a^{f^{-1}(x)} = x$ and $f^{-1}(a^x) = x$ become the following important statements:

Important Idea 10.2.1.
$a^{\log_a x} = x$ *AND* $\log_a(a^x) = x$

The notation for the logarithm function is rather strange and we will get back to it, but first let us look at some of the characteristics of the logarithm function. We can make some statements simply from what we already know about exponential functions and what we know about inverse functions in general. Recall that if two functions are inverses of each other then if one contains the ordered pair (x, y) then the other contains the ordered pair (y, x). This leads directly to some facts.

Important Idea 10.2.2.
The following are true concerning the logarithm function:

1. *The domain for* $\log_a x$ *is* $(0, \infty)$ *because* $(0, \infty)$ *is the range for* a^x.

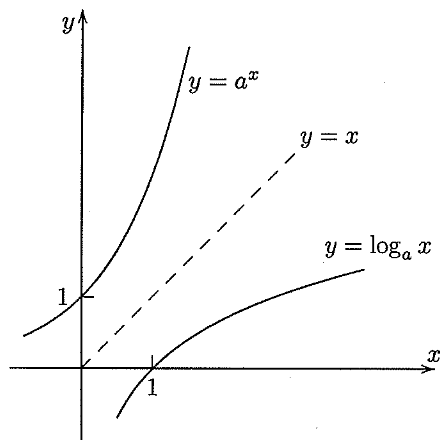
2. *The range for* $\log_a x$ *is* $\Re$ *because* $\Re$ *is the domain for* a^x.

3. $\log_a x$ *contains the point* $(1, 0)$ *because* a^x *contains the point* $(0, 1)$. *Thus, the logarithm function has an* x*-intercept, a root.*

4. $\log_a x$ *has no* y*-intercept because* a^x *has no* x*-intercept.*

5. $\log_a x$ *has a vertical asymptote* $x = 0$ *because* a^x *has a horizontal asymptote* $y = 0$.

The graph of the logarithm function is a reflection of the graph of the corresponding (same base) exponential function about the line $y = x$. Below we show the graphs for $y = a^x$ and $y = \log_a x$ when $a > 1$. It will be explained later why we do not need to be concerned separately for values of a between 0 and 1.



10.2.2 Getting a Grasp on Logarithmic Notation

Students often find the notation for logarithms to be a bit confusing. This section attempts to help you learn to work with it.

Important Idea 10.2.3.

$$\log_a x = y \text{ means } a^y = x$$

Learn this example to help you remember: $\log_2 8 = 3$ *means* $2^3 = 8$

The two equations, $\log_a x = y$ and $a^y = x$, mean exactly the same thing. They are two ways of expressing the same relationship between x and y. The first is called the *logarithmic* expression and the other is called the *exponential* expression.

Important Idea 10.2.4.

A logarithm IS an exponent.

In the expresssion above, the y is *equal* to the logarithm expression. The y is also the exponent in the exponential expression. This says that the value of a logarithm function represents the exponent. So, for example, $\log_{10} 1,000$ must equal 3 because 3 is the exponent where if 10 were the base the resulting value would be 1,000. $10^3 = 1,000$

Example 10.2.1.

1. $\log_3 9 = 2$ *because* $3^2 = 9$

2. $\log_5 1 = 0$ *because* $5^0 = 1$

3. $\log_2 \frac{1}{4} = -2$ *because* $2^{-2} = \frac{1}{4}$

4. $\log_e \sqrt{e} = \frac{1}{2}$ *because* $e^{\frac{1}{2}} = \sqrt{e}$

10.2.3 Common Logs, Natural Logs and Logs with base less than 1

Logarithms can be written with any base a where $0 < a < 1$ or $a > 1$, but two bases occur so frequently that we give these logarithms special names.

A common logarithm is a logarithm with a base of 10. It is fairly standard notation, and we will use if from this point on, that we do not bother to explicitly write the subscript "10" when we mean a common logarithm. So, when we write "$\log x$" we will mean "$\log_{10} x$".

A natural logarithm is a logarithm with a base of e. This logarithm is used extensively in calculus. The shortcut special notation for the natural logarithm is to just write "ln" instead of "$\log_e$". So, we write "$\ln x$" when we mean "$\log_e x$".

For all logarithms with bases different from 10 or e it is important that you write the base specifically as the function subscript. The equations $\log 100 = 2$ and $\ln(\frac{1}{e}) = -1$ are both correct. The equation $\log 8 = 3$ is not correct. One might have *meant* a base of 2 for this last equation but what was written is a common logarithm, and it certainly isn't true that $10^3 = 8$.

Let us now address the statement made earlier that while we can have logarithms with base between 0 and 1, we only need to concern ourselves with logarithms that have base $a > 1$. Any number between 0 and 1 can be written as $\frac{1}{a}$ where a is some number greater than 1. So, suppose we have $y = \log_{\frac{1}{a}} x$ This says the same thing as $(\frac{1}{a})^y = x$. This is the same thing as $a^{-y} = x$, which in turn is the same as $\log_a x = -y$. If we combine the first and last expressions, substituting for y, we get $\log_{\frac{1}{a}} x = -\log_a x$. So, anything that we need to do with logarithms that have a base between 0 and 1 we can do by using the reciprocal base (which is greater than 1) if we negate the logarithm. Actually, if you ponder this, it shouldn't be too surprising: Logarithms are exponents and if you negate the exponent of a number what happens?

Comprehension Check 10.2.

1. *Rewrite the following logarithmic equations into their equivalent exponential forms:*

   $\log_3 81 = 4$ $\quad \log\ .01 = -2$ $\quad \ln e^5 = 5$ $\quad \log_7 13 = x$

2. *Rewrite the following exponential equations into their equivalent logarithmic forms:*

   $2^{-1} = \frac{1}{2}$ $\quad e^{\frac{1}{3}} = \sqrt[3]{e}$ $\quad 10^2 = 100$ $\quad 3^{20} = x$

3. *To graphically illustrate the idea that* $\log_{\frac{1}{a}} x = -\log_a x$ *we will use* $a = 2$. *Sketch the following graphs:*

   (a) *Sketch* $y = \log_{\frac{1}{2}} x$ *by first drawing* $y = (\frac{1}{2})^x$, *and then reflecting this about the line* $y = x$ *to get the desired inverse* $y = \log_{\frac{1}{2}} x$.

   (b) *Sketch* $y = -\log_2 x$ *by first drawing* $y = \log_2 x$, *and then reflecting this about the* x*-axis to get* $y = -\log_2 x$.

   (c) *Compare your two resulting graphs. They should be the same.*

10.2.4 Properties of Logarithms

Logarithms are exponents, so the algebra associated with logarithms follows the same rules that one uses when dealing with exponents. However, the notation for logarithms can make this somewhat difficult to see. Often it is useful to rewrite the logarithmic expression into its exponential equivalent in order to understand the properties of logarithms. Several of the most important properties are listed next, followed by an explanation and examples for each one.

Properties of Logaithms

Here we assume that a, m, n, p are values consistent with the domain of the logarithm function. So, a, m, n, p are real numbers, a, m, n are positive and $a \neq 1$.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^p = p$
4. $a^{\log_a m} = m$
5. $\log_a mn = \log_a m + \log_a n$
6. $\log_a \frac{m}{n} = \log_a m - \log_a n$
7. $\log_a m^p = p \ \log_a m$

1. $\log_a 1 = 0$

    $\log_a 1 = 0$ because $a^0 = 1$

    Examples: $\log_5 1 = 0$ $\quad \ln 1 = 0$

2. $\log_a a = 1$

    $\log_a a = 1$ because $a^1 = a$

    Examples: $\log_2 2 = 1$ $\quad \ln e = 1$

3. $\log_a a^p = p$

    $\log_a a^p = p$ because $a^p = a^p$

    This is really just part of the original definition that the logarithm function is the inverse of the exponential function.

    Examples: $\log_4 16 = \log_4 4^2 = 2$ $\quad \ln \sqrt{e} = \frac{1}{2}$

    Notice that when $p = 1$ we just have the special case that is Property 2 above.

4. $a^{\log_a m} = m$

    Since $\log_a m$ represents "the exponent where if a were the base the resulting value would be m" it makes sense that if we use this $\log_a m$ as the exponent for a then our expression should be equal to m.

    This is really just the other part of the original definition that the logarithm function is the inverse of the exponential function.

    Examples: $6^{\log_6 17} = 17$ $\quad e^{\ln 4} = 4$

5. $\log_a mn = \log_a m + \log_a n$

    Here we will have to rewrite two logs into exponential form. So, we introduce names for them. Let $u = \log_a m$ and let $v = \log_a n$. This means that $a^u = m$ and $a^v = n$. So, the product mn is equal to $a^u a^v = a^{u+v}$. If we substitute a^{u+v} for mn in the original logarithm expression, we have $\log_a mn = \log_a a^{u+v}$ which is equal to $u + v$ (by Property 3). But $u + v$ is just $\log_a m + \log_a n$, so we have proven our claim that $\log_a mn = \log_a m + \log_a n$.

    Examples: $3 = \log_2 8 = \log_2(4 \cdot 2) = \log_2 4 + \log_2 2 = 2 + 1 = 3$

    $\log_4(3x) = \log_4 3 + \log_4 x$

6. $\log_a \frac{m}{n} = \log_a m - \log_a n$

This argument is left as an exercise for the student.

Examples: $3 = \log_2 8 = \log_2 \frac{32}{4} = \log_2 32 - \log_2 4 = 5 - 2 = 3$

$\log_5 \frac{2}{x} = \log_5 2 - \log_5 x$

7. $\log_a m^p = p\ \log_a m$

We will give an argument here for p a positive integer. We can give a rigorous proof for this property using an argument similar to that used for the two preceding properties. However, that argument is a tad more difficult to see and the integer argument has some instructional value so that is done here:

$$\log_a m^p = \log_a(\underbrace{m \cdot m \cdot \ldots \cdot m}_{p\ times}) = \underbrace{\log_a m + \log_a m + \ldots + \log_a m}_{p\ times} = p\ \log_a m$$

Examples: $6 = \log_2 64 = \log_2 8^2 = 2\log_2 8 = 2 \cdot 3 = 6$

$6 = \log_2 64 = \log_2 4^3 = 3\log_2 4 = 3 \cdot 2 = 6$

$\log_7 \sqrt[3]{x^2} = \log_7 (x)^{\frac{2}{3}} = \frac{2}{3}\log_7 x$

As is always a good idea, we have to be careful to only use the Properties when we have the proper domains for our functions. Sometimes this requires us to be alert. For example, we could easily misuse Property 7. If simply given the function $f(x) = \log_a x^2$ we would be tempted to say that it is equal to function $g(x) = 2\log_a x$. It is not. The domain of f is $\{x : x \neq 0\}$. The domain for g is $\{x : x > 0\}$. So, these two functions are not the same. Property 7 claims equality of the expressions because it restricts the domain, only allowing m to be positive.

That having been said, in all further examples and in the homework problems we will assume that the given variables are consistent with the domains of the Properties above.

We will now do a few examples where we take a single logarithmic expression and expand it into an equivalent expression that could use multiple logarithms.

Example 10.2.2.

$$\log_2(4x^2y) = \log_2 4 + \log_2 x^2 + \log_2 y = 2 + 2\log_2 x + \log_2 y$$

Example 10.2.3.

$$\ln\left(\frac{6}{\sqrt{x^2+1}}\right) = \ln 6 - \ln\sqrt{x^2+1} = \ln 6 - \frac{1}{2}\ln(x^2+1)$$

Example 10.2.4.

$$\log\left(\frac{x^2}{y^5z^3}\right)^4 = 4\log\left(\frac{x^2}{y^5z^3}\right) = 4(\log x^2 - \log(y^5z^3)) = 4(\log x^2 - \log y^5 - \log z^3)$$

$$= 4(2\log x - 5\log y - 3\log z) = 8\log x - 20\log y - 12\log z$$

Next are a few examples where we go the other way. We take a combination of logrithmic expressions and condense them into a single expression with a coefficient of 1.

Example 10.2.5.

$$\log_3(x+2y) - \log_3(x-y) = \log_3 \frac{x+2y}{x-y}$$

Example 10.2.6.

$$\log x^2 + \frac{1}{2}\log y - \log z = \log x^2 + \log\sqrt{y} - \log z = \log \frac{x^2\sqrt{y}}{z}$$

Example 10.2.7.

$$\frac{1}{3}(\ln x - 2\ln y) + 5\ln z = \frac{1}{3}\ln\left(\frac{x}{y^2}\right) + \ln z^5$$

$$= \ln\sqrt[3]{\frac{x}{y^2}} + \ln z^5 = \ln\left(z^5\sqrt[3]{\frac{x}{y^2}}\right)$$

When using the Properties to combine or expand logarithms be careful to obey the usual rules of algebra and order of operation. Look at the following set of expressions. Make sure you understand how they are different.

Example 10.2.8.

1.

$$\frac{1}{2}\log x + \log y - \log(2z) = \log x^{\frac{1}{2}} + \log y - \log(2z)$$

$$= \log\sqrt{x} + \log y - \log(2z) = \log\frac{\sqrt{x}y}{2z}$$

2.

$$\frac{1}{2}(\log x + \log y) - \log(2z) = \frac{1}{2}\log(xy) - \log(2z)$$

$$= \log(xy)^{\frac{1}{2}} - \log(2z) = \log\sqrt{xy} - \log(2z) = \log\frac{\sqrt{xy}}{2z}$$

Or, you could use the distributive property on the $\frac{1}{2}$:

$$\frac{1}{2}(\log x + \log y) - \log(2z) = \frac{1}{2}\log x + \frac{1}{2}\log y - \log(2z)$$

$$= \log x^{\frac{1}{2}} + \log y^{\frac{1}{2}} - \log(2z) = \log\sqrt{x} + \log\sqrt{y} - \log(2z)$$

$$= \log\frac{\sqrt{x}\sqrt{y}}{2z} = \log\frac{\sqrt{xy}}{2z}$$

3.

$$\frac{1}{2}(\log x + \log y - \log(2z)) = \frac{1}{2}\left(\log\frac{xy}{2z}\right)$$

$$= \log\left(\frac{xy}{2z}\right)^{\frac{1}{2}} = \log\sqrt{\frac{xy}{2z}}$$

We can use the Properties of Logarithms to rewrite and simplify logarithmic expressions.

Example 10.2.9.

$$\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6 36 = 2$$

Example 10.2.10.

$$\log_9 25 - \log_9 75 = \log_9 \frac{25}{75} = \log_9 \frac{1}{3} = -\frac{1}{2}$$

Example 10.2.11.

$$\frac{2}{3}\ln 27 + 2\ln 2 - \ln 3 = \ln 27^{\frac{2}{3}} + \ln 2^2 - \ln 3$$

$$= \ln 9 + \ln 4 - \ln 3 = \ln \frac{9 \cdot 4}{3} = \ln 12$$

Comprehension Check 10.3.

1. *Explain in words the difference between* $(\log_a x)(\log_a y)$ *and* $\log_a(xy)$. *Which one is used in Property 5?*

2. *Explain in words the difference between* $\dfrac{\log_a x}{\log_a y}$ *and* $\log_a\left(\dfrac{x}{y}\right)$. *Which one is used in Property 6?*

3. *Explain in words the difference between* $(\log_a x^p)$ *and* $(\log_a x)^p$. *Which one is used in Property 7?*

10.2.5 Changing Bases

Suppose you would like to know the approximate value of x for $2^x = 100$. You know that the number is somewhere between 6 and 7 (why)?, but you need to be more precise than that. You are looking for an exponent value. So, you are looking for a logarithm. In particular, you want to know the value of $\log_2 100$. This is good so far. But when you then go to your calculator you realize that it doesn't handle logarithms with a base of 2. The only logarithm buttons on your calculator are "log" (common logs, base 10) and "ln" (natural logs, base e). You need to be able to change a base 2 logarithm into a base 10 or base e logarithm.

There is a straightforward way to change from one base to another. It uses the algebra that we already know for logarithms, and the following fact:

Important Idea 10.2.5.

If $x = y$, *then* $\log_a x = \log_a y$ *AND* *If* $\log_a x = \log_a y$, *then* $x = y$.

Follow the steps as we change from a base a logarithm $\log_a x$ into an expression involving base b logarithms.

We will call our base a logarithm y. So,

$$y = \log_a x$$
$$a^y = x$$
$$\log_b a^y = \log_b x$$
$$y \log_b a = \log_b x$$
$$y = \frac{\log_b x}{\log_b a}$$

Now, substituting back for y we get:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

This is the "Change of Base Formula for Logarithms", restated below.

Change of Base Formula for Logarithms

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Look carefully at the placement of the a's, b's, x's.

For our particular example we can change $\log_2 100$ into $\frac{\log 100}{\log 2}$. A calculator will give approximate values: $\frac{2}{0.30103} \approx 6.64386$

We could also change to natural logs: $\log_2 100 = \frac{\ln 100}{\ln 2} \approx \frac{4.60517}{0.69315} \approx 6.64386$.

The intermediate values in these calculations were different but the end result was the same, and was indeed a number between 6 and 7.

In the example above we found it useful to change a base 2 logarithm into a common log or natural log. However, the change of base formula works for changing to any base. $\log_2 100$ is in fact equal to $\frac{\log_7 100}{\log_7 2}$, $\frac{\log_{13} 100}{\log_{13} 2}$, and $\frac{\log_{88} 100}{\log_{88} 2}$. All of these fractions are approximately 6.64386.

Example 10.2.12.
Change $\log_3 25$ *into a base 7 logarithm.*

Using the Change of Base for Logarithms formula we apply: $a = 3, b = 7$ *and* $x = 25$

$$\log_3 25 = \frac{\log_7 25}{\log_7 3}$$

Example 10.2.13.
Show that $\ln 10 = \frac{1}{\log e}$.

$$\ln 10 = \frac{\log 10}{\log e} = \frac{1}{\log e}$$

It would be reasonable to ask if there is a change of base formula for exponential expressions. In other words, if we are given the expression a^x is there some y so that $a^x = b^y$ for a desired base b? Yes. We find it by simply solving the equation $a^x = b^y$ for y.

$$a^x = b^y$$

$$\log_b a^x = \log_b b^y$$

$$x \log_b a = y$$

Thus we have the following result:

Change of Base Formula for Exponential Expressions

$$a^x = b^{x \log_b a}$$

Look carefully at the placement of the a's, b's, x's.

Since most calculators can handle expressions of various bases, the change of base formula for exponential expressions is not as critical as the change of base formula for logarithms. However, there is a great need in calculus to be able to change the base of an exponential expression to an equivalent expression that uses the base e. The general formula above becomes:

$$a^x = e^{x \ln a}$$

Example 10.2.14.
Change the function $f(x) = 2^x$ to an equivalent function that uses e as its base.

$$f(x) = 2^x = e^{\ln 2^x} = e^{x \ln 2}$$

10.2.6 Solving Logarithmic and Exponential Equations

We can use the idea that logarithm functions and exponential functions are inverses when we try to solve equations that include these functions. Just like when we use subtraction to undo addition, and we use multiplication to undo division and we use cosine to undo arccosine we will now use logarithms to undo exponential functions and vice-versa. We will start with some examples of using logarithms to solve exponential functions. In all examples we will solve for x.

Example 10.2.15.
Solve for x: $y = 3e^x$

$$y = 3e^x. \text{ So, } \left(\tfrac{y}{3}\right) = e^x.$$

Now we "take the natural log" of both sides: $\ln\left(\frac{y}{3}\right) = \ln e^x$.

But $\ln e^x = x$ *so we are finished.* $\ln\left(\frac{y}{3}\right) = x$

Notice that we used a logarithm with base e in order to solve for an x that was in an exponential expression with base e. We used the logarithm to bring the x out of the exponent. Sometimes we use this operation without really realizing it. We can mentally do the rewriting of the logarithm into its equivalent exponential form. We are still using the logarithm concept, just not writing it down or even acknowledging it. This point is illustrated in Example 10.2.16. Compare both problems in this example. Where in the first problem are we implicitly using the logarithm?

Example 10.2.16.
Solve for x: $2(1 + 4^x) = 6$ *and* $2(1 + 4^x) = 12$

$$\begin{aligned} 2(1+4^x) &= 6 \\ 1+4^x &= 3 \\ 4^x &= 2 \\ x &= \tfrac{1}{2} \end{aligned} \qquad \begin{aligned} 2(1+4^x) &= 12 \\ 1+4^x &= 6 \\ 4^x &= 5 \\ \log_4 4^x &= \log_4 5 \\ x &= \log_4 5 \end{aligned}$$

Notice that in the second problem of Example 10.2.16 we leave our answer in terms of a base 4 logarithm. While that is correct, it is not very useful for giving us an idea of the value of the answer. In Example 10.2.17 we show three ways of solving an exponential equation. All of the answers are equal. The prefered method will depend on the desired application and the capabilities of your calculator.

Example 10.2.17.
Solve for x: $8 = 4(3^{2x+1})$

$$\begin{aligned} 8 &= 4(3^{2x+1}) \\ 2 &= 3^{2x+1} \\ \log_3 2 &= \log_3(3^{2x+1}) \\ \log_3 2 &= 2x+1 \\ \log_3 2 - 1 &= 2x \\ \frac{\log_3 2 - 1}{2} &= x \end{aligned} \qquad \begin{aligned} 8 &= 4(3^{2x+1}) \\ 2 &= 3^{2x+1} \\ \log 2 &= \log(3^{2x+1}) \\ \log 2 &= (2x+1)\log 3 \\ \frac{\log 2}{\log 3} &= 2x+1 \\ \frac{\log 2}{\log 3} - 1 &= 2x \\ \frac{\frac{\log 2}{\log 3} - 1}{2} &= x \end{aligned} \qquad \begin{aligned} 8 &= 4(3^{2x+1}) \\ 2 &= 3^{2x+1} \\ \ln 2 &= \ln(3^{2x+1}) \\ \ln 2 &= (2x+1)\ln 3 \\ \frac{\ln 2}{\ln 3} &= 2x+1 \\ \frac{\ln 2}{\ln 3} - 1 &= 2x \\ \frac{\frac{\ln 2}{\ln 3} - 1}{2} &= x \end{aligned}$$

When you "take the log" of both sides of an equation you must use the same-based log on both sides. If you have an exponential equation with multiple bases then you need to decide which logarithm base you wish to use. You could choose any of the bases that are in the problem or you could choose something completely different, such as a common log or a natural log. Again, we offer an example with multiple, but equivalent, solutions.

Example 10.2.18.
Solve for x: $5^x = 6^{x-1}$

$$\begin{aligned} 5^x &= 6^{x-1} \\ \log_5 5^x &= \log_5 6^{x-1} \\ x &= (x-1)\log_5 6 \\ x &= x\log_5 6 - \log_5 6 \\ x - x\log_5 6 &= -\log_5 6 \\ x(1-\log_5 6) &= -\log_5 6 \\ x &= \frac{-\log_5 6}{1-\log_5 6} \end{aligned} \qquad \begin{aligned} 5^x &= 6^{x-1} \\ \log_6 5^x &= \log_6 6^{x-1} \\ x\log_6 5 &= x-1 \\ x\log_6 5 - x &= -1 \\ x(\log_6 5 - 1) &= -1 \\ x &= \frac{-1}{\log_6 5 - 1} \end{aligned} \qquad \begin{aligned} 5^x &= 6^{x-1} \\ \ln 5^x &= \ln 6^{x-1} \\ x\ln 5 &= (x-1)\ln 6 \\ x\ln 5 &= x\ln 6 - \ln 6 \\ x\ln 5 - x\ln 6 &= -\ln 6 \\ x(\ln 5 - \ln 6) &= -\ln 6 \\ x &= \frac{-\ln 6}{\ln 5 - \ln 6} \end{aligned}$$

Finally, we have an example that reminds us that we can still use all of the creativity of algebra. We are simply now including the logarithm to help us with the exponential pieces of equations.

Example 10.2.19.
Solve for x: $2(3^{2x}) - 3^x - 3 = 0$

$$2(3^{2x}) - 3^x - 3 = 0$$

$$2(3^x)^2 - (3^x) - 3 = 0$$

Now let "u" = (3^x)

$$2u^2 - u - 3 = 0$$
$$(2u-3)(u+1) = 0$$
$$u = \tfrac{3}{2} \quad or \quad u = -1$$
$$3^x = \tfrac{3}{2} \quad or \quad 3^x = -1$$

$x = \log_3 \frac{3}{2}$ *is the only solution because* $3^x = -1$ *has no solution.*

We now start with some logarithmic equations and will use the inverse operation of raising the expression to a power in order to solve for x.

Example 10.2.20.
Solve for x: $\log_5(x+3) = 2$

$$\log_5(x+3) = 2$$

$$5^{\log_5(x+3)} = 5^2$$
$$x + 3 = 25$$
$$x = 22$$

Notice that the base we choose for the "raising both sides" is the same as the logarithm base.

Example 10.2.21.
Solve for x: $2 + 3\ln(x-10) = 0$

$$2 + 3\ln(x-10) = 0$$

$$3\ln(x-10) = -2$$
$$\ln(x-10) = \frac{-2}{3}$$
$$e^{\ln(x-10)} = e^{\frac{-2}{3}}$$
$$x - 10 = e^{\frac{-2}{3}}$$
$$x = e^{\frac{-2}{3}} + 10$$

Notice that we isolate the logarithm before applying the "raising both sides" operation. If you have more than one logarithmic expression you should combine them first.

Example 10.2.22.
Solve for x: $\log_3 x + \log_3(x+2) = 1$

$$\log_3 x + \log_3(x+2) = 1$$

$$\log_3(x \cdot (x+2)) = 1$$
$$\log_3(x^2 + 2x) = 1$$
$$3^{\log_3(x^2+2x)} = 3^1$$
$$x^2 + 2x = 3$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$

$x = -3$ and $x = 1$ appear to be solutions. However, $x = -3$ is not a solution because it is not in the domain of the original problem.

Important Idea 10.2.6.
When solving logarithmic equations always check your solution in the original problem to make sure that you have no domain violations.
Remember that the domain for a logarithm function can only be values that make its argument positive.

In Example 10.2.22 neither $\log_3 x$ nor $\log_3(x+2)$ can accept $x = -3$ as input. However, you only need to have one given logarithmic expression undefined by your "solution" to make that "solution" invalid.

Go back to Examples 10.2.20 and 10.2.21 and make sure that the solutions presented are valid.

10.3 Exercises

Problems for Section 10.1

Problem 1. Use the methods of Section 10.1 to solve each equation for x.

(a) $2(1+4^x) = 6$ (b) $2^{x+3} = 4^{x-1}$ (c) $\dfrac{5^{x+3}}{5^{2x}} = 25$ (d) $x^3e^x - e^x = 0$

Problem 2. Without using a table of values and plotting points, sketch the graph of each of the following exponential functions. On your graph label the y-intercepts, and give the equation of the asymptote.

(a) $f(x) = 2^x - 3$ (b) $f(x) = \left(\frac{1}{e}\right)^x$ (c) $f(x) = -3^x + 1$ (d) $f(x) = 5^{-x}$

Problem 3. For what values of x is $\left(\frac{1}{2}\right)^x < \left(\frac{1}{3}\right)^x$? Use graphs to help you decide.

Problems for Section 10.2

Problem 1. Change each logarithmic statement to its equivalent exponential form.

(a) $5 = \log_2 x$ (b) $y = \ln 27$ (c) $12 = \log_a 5$

Problem 2. Change each exponential statement to its logarithmic equivalent form.

(a) $3^x = 2$ (b) $e^5 = y$ (c) $x^4 = 9$

Problem 3. Evaluate the following numbers without using a calculator.

(a) $\ln 1$ (b) $\log(.01)$ (c) $\log_3 81$

(d) $\log_4 2$ (e) $\log_3 \frac{1}{27}$ (f) $\log_{\frac{1}{2}} 8$

(g) $\log_4 2 + \log_4 16$ (h) $\log_2 \sqrt[3]{\sqrt[4]{2}}$ (i) $2\ln e^3 + \ln e^{-5} + e^{\ln 3}$

Problem 4. Without using a calculator, find the value of x.

(a) $\log_9 x = \frac{1}{2}$ (b) $\log_x 11 = 1$ (c) $\log_x 27 = \frac{3}{2}$ (d) $\log_6 x = -2$

Problem 5. Which is larger: $\log_6 37$ or $\log_7 48$? Why? Do not use a calculator.

Problem 6. Without making a table of values and plotting points, sketch the graph of each of the following logarithmic functions. Give the equation of the asymptote.

(a) $f(x) = \log_2 x + 3$ (b) $f(x) = \log_2(x+3)$ (c) $f(x) = -\ln x$

Problem 7. Find the domain for each of the following functions.

(a) $f(x) = \log_3(x-7) - \log_3(x+2)$ (b) $f(x) = \log 3^x$

(c) $f(x) = \ln(e - x)$ (d) $f(x) = \ln(x^2 - x - 2)$

Problem 8. Given $f(x) = \log_3(x+2)$ find its domain, find its inverse $f^{-1}(x)$, and give the range of $f^{-1}(x)$.

Problem 9. Use the Properties of Logarithms to condense each of these expressions into a single logarithmic expression with a positive exponent and a coefficient of 1.

(a) $\log_3 x + \log_3 2$ (b) $\log_2 9 - \log_2 y$ (c) $2\log x - 5\log y$

(d) $\frac{1}{2}\log_4(x+5)$ (e) $-4\log_6(2x)$ (f) $3\ln x + 4\ln y - 4\ln z$

(g) $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

Problem 10. Use the Properties of Logarithms to expand each of these single logarithms into expressions with multiple logarithms having single character arguments.

(a) $\log_3\left(\frac{y}{2}\right)$ (b) $\log(10x)$ (c) $\log_6\left(\frac{1}{z^3}\right)$ (d) $\log_4(4x^2y)$

(e) $\log_4(4xy)^2$ (f) $\log\left(\frac{x^2-1}{x^3}\right)$ (g) $\ln\sqrt[5]{\frac{x^2}{y^3}}$ (h) $\log_2\frac{\sqrt{x}}{z^4}$

Problem 11. Use the Properties of Logarithms to evaluate the following:

(a) $\log_6 12 + \log_6 3 - \ln 1$ (b) $\frac{2}{3}\log_4 8 + \frac{1}{2}\log_4 9 - \log_4 6$

Problem 12.

(a) Use Property 6 to show that $\log_a \frac{1}{n} = -\log_a n$

(b) Use Property 7 to show that $\log_a \frac{1}{n} = -\log_a n$

Problem 13. Prove Property 6. Use the argument done for Property 5 as a model and make the necessary adjustments. (C'mon, do it! It's not bad).

Problem 14. In Example 10.2.13 we showed that $\ln 10 = \dfrac{1}{\log e}$.

(a) How large would you expect the number $\ln 10$ to be (make a reasonable guess, remembering that you are looking for an exponent which will give you 10 if e is the base)?

(b) How large would you expect the number $\log e$ to be (make a reasonable guess, remembering that you are looking for an exponent which will give you e if 10 is the base)?

(c) Use a calculator to find the approximate values for $\ln 10$ and $\log e$. How close were your estimates?

(d) Check (still using your calculator) that these numbers are indeed reciprocals).

Problem 15. Rewrite $\log_7 9$ as an equivalent logarithm using:

(a) base 5 (b) base 10 (c) base e

Problem 16. Use the Change of Base Formula to show that $\log_a b = \dfrac{1}{\log_b a}$

Problem 17. Use the Change of Base Formula to show that $\log_2 5 = 2\log_4 5$

Problem 18. Use the Change of Base for Exponential Expressions to change each expression below to its equivalent expression in base e.

(a) 3^x (b) 6^x (c) 10^x

Problem 19. Given: $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.10$ and $\ln 5 \approx 1.61$, find approximate values for the following numbers without using a calculator:

(a) $\ln 15$ (b) $\ln(1.5)$ (c) $\ln \sqrt[3]{2}$ (d) $\ln(0.9)$

Problem 20. Given: $\log 3 \approx 0.477$, find approximate values for the following numbers without using a calculator:

(a) $\log 30$ (b) $\log 3000$ (c) $\log_3 10$

Problem 21. Given the number $\log_9 21$:

(a) Rewrite it as a ratio of common logs
(b) Rewrite it as a ratio of natural logs
(c) Show that it is equal to $\log_3 \sqrt{21}$

Problem 22. Solve the following exponential equations for x

(a) $2^{x-3} = 32$
(b) $2(5^x) = 32$
(c) $6^x + 10 = 47$
(d) $7(4^{6x-2}) + 13 = 41$
(e) $3^{2x} - 5(3^x) - 6 = 0$
(f) $4^x = 3^{2x-1}$ Use a natural log
(g) $2^x = 2(5^x)$ Use a common log

Problem 23. Solve the following logarithmic equations for x

(a) $2\log_5(3x) = 4$
(b) $3 + 2\log x = 15$
(c) $\ln(\ln x) = 2$
(d) $\log_3(x+1) - \log_3 x = 2$
(e) $\log_{12}(x-6) - \log_{12} 4 = 1$
(f) $\log_6(x+2) + \log_6(x+7) = 2$
(g) $\log_3(2x-1) = 2\log_3 x$

10.4 Answers to Exercises

Answers for Section 10.1 Exercises

Answer to Problem 1.

(a) $\frac{1}{2}$ (b) 5 (c) 1 (d) 1

Answer to Problem 2.

(a) $(0,-2),\ y=-3$ (b) $(0,1),\ y=0$ (c) $(0,0),\ y=1$ (d) $(0,1),\ y=0$

Answer to Problem 3.

$x<0$

Answers for Section 10.2 Exercises

Answer to Problem 1.

(a) $2^5=x$ (b) $e^y=27$ (c) $a^{12}=5$

Answer to Problem 2.

(a) $\log_3 2=x$ (b) $\ln y=5$ (c) $\log_x 9=4$

Answer to Problem 3.

(a) 0 (b) -2 (c) 4 (d) $\dfrac{1}{2}$ (e) -3

(f) -3 (g) $\dfrac{5}{2}$ (h) $\dfrac{1}{12}$ (i) 4

Answer to Problem 4.

(a) 3 (b) 11 (c) 9 (d) $\dfrac{1}{36}$

Answer to Problem 5.

$\log_6 37$

Answer to Problem 6.

(a) $x=0$ (b) $x=-3$ (c) $x=0$

Answer to Problem 7.

(a) $\{x\in\Re : x>7\}$ (b) $\Re$ (c) $\{x\in\Re : x<e\}$ (d) $\{x\in\Re : x<-1 \text{ or } x>2\}$

Answer to Problem 8.

$D_f = (-2, \infty) \qquad f^{-1}(x) = 3^x - 2 \qquad R_{f^{-1}} = D_f = (-2, \infty).$

Answer to Problem 9.

(a) $\log_3(2x)$ (b) $\log_2\left(\frac{9}{y}\right)$ (c) $\log\left(\frac{x^2}{y^5}\right)$ (d) $\log_4\sqrt{x+5}$

(e) $\log_6\left(\frac{1}{16x^4}\right)$ (f) $\ln\left(\frac{x^3y^4}{z^4}\right)$ (g) $\log_2\sqrt[3]{x^2+x}$

Answer to Problem 10.

(a) $\log_3 y - \log_3 2$ (b) $1 + \log x$ (c) $-3\log_6 z$

(d) $1 + 2\log_4 x + \log_4 y$ (e) $2 + 2\log_4 x + 2\log_4 y$ (f) $\log(x^2-1) - 3\log x$

(g) $\frac{2}{5}\ln x - \frac{3}{5}\ln y$ (h) $\frac{1}{2}\log_2 x - 4\log_2 z$

Answer to Problem 11.

(a) 2 (b) $\frac{1}{2}$

Answer to Problem 12.

(a) $\log_a\left(\frac{1}{n}\right) = \log_a 1 - \log_a n = 0 - \log_a n = -\log_a n$

(b) $\log_a\left(\frac{1}{n}\right) = \log_a n^{-1} = -1\log_a n = -\log_a n$

Answer to Problem 13.
Proof not shown.

Answer to Problem 14.
Answers will vary.

Answer to Problem 15.

(a) $\frac{\log_5 9}{\log_5 7}$ (b) $\frac{\log_{10} 9}{\log_{10} 7}$ (c) $\frac{\ln 9}{\ln 7}$

Answer to Problem 16.
$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$

Answer to Problem 17.
$\log_2 5 = \frac{\log_4 5}{\log_4 2} = \frac{\log_4 5}{\frac{1}{2}} = 2\log_4 5$

Answer to Problem 18.

(a) $e^{x\ln 3}$ (b) $e^{x\ln 6}$ (c) $e^{x\ln 10}$

Answer to Problem 19.

(a) 2.71 (b) 0.41 (c) 0.23 (d) -0.1

Answer to Problem 20.

(a) 1.477 (b) 3.477 (c) $\dfrac{1}{0.477}$

Answer to Problem 21.

(a) $\dfrac{\log 21}{\log 9}$ (b) $\dfrac{\ln 21}{\ln 9}$ (c) Proof not shown. Hint: Rewrite using base 3.

Answer to Problem 22.

(a) 8 (b) $\log_5 16$ (c) $\log_6 37$ (d) $\dfrac{1}{2}$

(e) $\log_3 6$ (f) $\dfrac{-\ln 3}{\ln 4 - 2\ln 3}$ (g) $\dfrac{\log 2}{\log 2 - \log 5}$

Answer to Problem 23.

(a) $\dfrac{25}{3}$ (b) $1,000,000$ (c) e^{e^2} (d) $\dfrac{1}{8}$ (e) 54 (f) 2 (g) 1