Section 26, #8 (Homework problem)

$f(x,y) = x^4 + y^3 + \frac{3}{xy}$ Find relative extrema.

Note: $x \neq 0$, $y \neq 0$

$f_x = 4x^3 + \frac{3}{y}(-x^{-2}) = 4x^3 - \frac{3}{yx^2}$

$f_y = 3y^2 - \frac{3}{xy^2}$

$f_y = 0 = 3y^2 - \frac{3}{xy^2}$

$0 = 3y^{-2}(y^4 - \frac{1}{x}) \Rightarrow y^4 = \frac{1}{x}$

$f_x = 0 = 4x^3 - \frac{3}{yx^2}$

$0 = 4x^3 - \frac{3}{y}\left(\frac{1}{x}\right)^2$

$0 = 4x^3 - \frac{3}{y}(y^4)^2$

$0 = 4x^3 - 3y^7 \Rightarrow 4x^3 = 3y^7$

$4x^3 = 3y^4y^3$

$4x^3 = 3\left(\frac{1}{x}\right)y^3$

$4x^4 = 3y^3$

$y^4 = \frac{1}{x} \Rightarrow x = \frac{1}{y^4} \longrightarrow 4\left(\frac{1}{y^4}\right)^4 = 3y^3$

$\frac{4}{y^{16}} = 3y^3$

$4 = 3y^{19}$

$\frac{4}{3} = y^{19}$

$\sqrt[19]{\frac{4}{3}} = y$

$x = \frac{1}{\sqrt[19]{\left(\frac{4}{3}\right)^4}}$

$x = \sqrt[19]{\left(\frac{3}{4}\right)^4}$

crit. pt. $\left(\sqrt[19]{\left(\frac{3}{4}\right)^4}, \sqrt[19]{\frac{4}{3}}\right)$

Note: x, y are both positive

$f_x = 4x^3 - \frac{3}{yx^2}$

$f_{xx} = 12x^2 - \frac{3}{y}(-2x^{-3})$

$= 12x^2 + \frac{6}{yx^3}$

$f_{xy} = -\frac{3}{x^2}(-y^{-2}) = \frac{3}{x^2y^2}$

$f_y = 3y^2 - \frac{3}{xy^2}$

$f_{yy} = 6y + \frac{6}{xy^3}$

Check:
$f_{yx} = -\frac{3}{y^2}(-x^{-2}) = \frac{3}{x^2y^2}$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= \left(12x^2 + \frac{6}{yx^3}\right)\left(6y + \frac{6}{xy^3}\right) - \left(\frac{3}{x^2y^2}\right)^2$$

$$= \left[72x^2y + \frac{72x}{y^3} + \frac{36}{x^3} + \underbrace{\frac{36}{x^4y^4}\Big] - \frac{9}{x^4y^4}}_{\text{like terms}}$$

$$D = 72x^2y + \frac{72x}{y^3} + \frac{36}{x^3} + \frac{27}{x^4y^4}$$

Since x, y are both positive, $D > 0$.

AND $f_{xx} = 12x^2 + \frac{6}{yx^3} > 0$

So, ... (finally!) we have a local minimum at

$$\left(\sqrt[19]{\left(\frac{3}{4}\right)^4}, \sqrt[19]{\frac{4}{3}}\right)$$