this property. [We will show elsewhere that the simple FP_∞ group mentioned above also has this property.]

The paper is organized as follows. In §1 we summarize known properties of F, emphasizing the point of view that F admits the universal example of an endomorphism which is idempotent up to conjugacy. In §2 we give a heuristic discussion of the topological analogue of F; this is a space Y admitting the universal example of a self map which is idempotent up to homotopy. The rigorous construction of Y is given in §3. We obtain Y as X/F, where X is a space with a free right F-action. It is possible to read §3 independently of §2, but the construction of X will then seem very strange.

We prove in §4 that X is contractible, so that Y is an Eilenberg-MacLane space $K(F,1)$. In §5 we show that Y has the homotopy type of a complex Z of finite type, i.e., with only finitely many cells in each dimension. This proves that F is of type FP_∞. In fact, the cellular chain complex P of the universal cover of Z is a finite type free resolution of $\mathbf{Z}$ over $\mathbf{Z}F$.

We give in §6 a purely algebraic description of this chain complex P. This *description* can be read independently of the rest of the paper. But the proof that P is in fact a resolution relies on the considerations of §§3–5, and we know of no way to avoid this.

In §7 we use the results of the previous sections to calculate $H_*(F,\mathbf{Z})$ and $H^*(F,\mathbf{Z}F)$. This second calculation depends on a result in our paper [BG_2]. Finally, we give in §8 an application of our work to the theory of homotopy idempotents.

We wish to acknowledge our debt to Jerzy Dydak. In unpublished joint work of Dydak and one of us (R.G.) in 1980, there appears a complex Z' of finite type very similar to the space Z mentioned above. It was conjectured at that time that Z' was a $K(F,1)$. The construction of Z' motivated the present work. (In fact, it follows from the work here that Z' *is* a $K(F,1)$.)

This paper and [BG_2] together contain proofs of all theorems announced in [BG_1].

Notational conventions

Given elements a, b of a group, we set $a^b = b^{-1}ab$ and $[a,b] = aba^{-1}b^{-1}$.

All group actions in this paper will be *right* actions. In particular, if G is the group of homeomorphisms of a space X, then the action of G on X is denoted $(x,g) \mapsto xg\,(x \in X, g \in G)$, and composition in G is defined by $x(gh) = (xg)h\,(x \in X, g, h \in G)$.

Given a space Y with basepoint, the composition law in $\pi_1(Y)$ is defined by $[\omega]\cdot[\omega'] = [\omega \circ \omega']$, where $\omega \circ \omega'$ traverses ω' followed by ω. This is consistent with the convention of the previous paragraph, in the following sense: Suppose X is the universal cover of Y and G is the group of deck transformations. Then, under the usual hypotheses of covering space theory, there is an isomorphism $\pi_1(Y) \xrightarrow{\approx} G$ which sends $[\omega]$ to the element $g \in G$ such that the lift of ω starting at the basepoint x_0 of X ends at $x_0 g$.

1. The group F

Everything in this section can be found in one or more of [MT], [FH], [D$_1$], [D$_2$], [DH], [DS]. For the convenience of the reader, we have included proofs of all non-obvious results which will be needed later in the paper.

Recall that F is defined by the presentation

$$\langle x_0, x_1, x_2, \ldots | x_n^{x_i} = x_{n+1} \text{ for } i<n \rangle.$$

It is immediate from this definition that F admits a "shift" map ϕ: $F \to F$ such that $\phi(x_n) = x_{n+1}$ for $n \geqq 0$ and such that $\phi^2(x) = \phi(x)^{x_0}$ for all $x \in F$; thus ϕ is *idempotent up to conjugacy*. Moreover, the triple (F, ϕ, x_0) is the *universal example* of this situation:

Proposition 1.1. *Given a group G, an endomorphism ψ: $G \to G$, and an element $g_0 \in G$ such that $\psi^2(g) = \psi(g)^{g_0}$ for all $g \in G$, there is a unique homomorphism f: $F \to G$ such that $f(x_0) = g_0$ and $f\phi = \psi f$.* □

Remark. Let $g_n = \psi^n(g_0)$ in the situation of 1.1. If $g_1 \neq g_2$, it can be shown that f: $F \to G$ is injective, so that the g_n generate a copy of F. More generally, it is known that F admits no nonabelian proper quotients.

Example 1.2. Let g_n $(n \geqq 0)$ be the piecewise linear homeomorphism of $\mathbf{R}$ which is the identity on $(-\infty, n]$, has slope 2 on $[n, n+1]$, and has slope 1 on $[n+1, \infty)$. Let S be the homeomorphism of $\mathbf{R}$ given by $uS = u+1$ $(u \in \mathbf{R})$, and note that $g_n^S = g_{n+1}$. The group of homeomorphisms G generated by the g_n therefore admits an endomorphism ψ given by $\psi(g) = g^S$ and satisfying $\psi(g_n) = g_{n+1}$. Note that g_0 agrees with S on $[1, \infty)$; since $\psi(g)$ has support in $[1, \infty)$ for every $g \in G$, it follows that $\psi^2(g) = \psi(g)^S = \psi(g)^{g_0}$. The proposition therefore yields a homomorphism f: $F \to G$ such that $f(x_n) = g_n$. [In view of the remark above, f is in fact an isomorphism. But an independent proof of this fact will be given below.]

We will use this homomorphism f to deduce a number of properties of F.

(1.3) **Normal forms.** The relations defining F allow one to write any $x \in F$ in the form $x = x_{i_1} \ldots x_{i_k} x_{j_m}^{-1} \ldots x_{j_1}^{-1}$ with $i_1 \leqq \ldots \leqq i_k$, $j_1 \leqq \ldots \leqq j_m$, k, $m \geqq 0$. Moreover, we can choose this expression for x so that if x_i and x_i^{-1} both occur for some i then x_{i+1} or x_{i+1}^{-1} also occurs. [Otherwise there would be a subproduct of the form $x_i \phi^{i+2}(y) x_i^{-1}$, which could be replaced by $\phi^{i+1}(y)$.] An expression of this form is called a *normal form* for x. We claim that x admits a *unique* normal form.

Proof. It suffices to show that any $g \in G$ has a unique normal form in terms of the g_n. Suppose first that g has a normal form $g_{i_1} \ldots g_{i_k} g_{j_m}^{-1} \ldots g_{j_1}^{-1}$ as above, with all subscripts $\geqq$ some integer i. Then the right-hand derivative of g at $u = i$ is 2^n, where n is the g_i-exponent sum in the normal form. In particular, any other normal form for g with subscripts $\geqq i$ will have the same g_i-exponent sum.

Suppose now that there are two different normal forms for the same element of G, and choose such a pair of normal forms of minimal total length. That total length is necessarily >0. Let i be the smallest subscript occurring in either normal form. The normal forms cannot both start with g_i, since we could then cancel g_i, contradicting the minimality. Similarly, they cannot both end with g_i^{-1}. Since they have the same g_i-exponent sum by the previous paragraph, the only possibility is that one of the normal forms involves both g_i and g_i^{-1} and the other involves neither. The equality between the homeomorphisms represented by the normal forms therefore reads $g_i z g_i^{-1} = w$, where z is a normal form with subscripts $\geqq i$ and w is a normal form with subscripts $\geqq i+1$. Then $z = w^{g_i} = \psi(w)$ as homeomorphisms, hence also as formal expressions by the minimality of our supposed counter-example. But then z involves only subscripts $\geqq i+2$, contradicting the fact that the expression $g_i z g_i^{-1}$ is a normal form. □

As a corollary of the proof, we have:

Corollary 1.4. *The homomorphism $f\colon F \to G$ of* 1.2 *is an isomorphism.* □

The group G is torsion-free, and $\psi\colon G \to G$ is injective. Hence:

Corollary 1.5. *F is torsion-free.* □

Corollary 1.6. *$\phi\colon F \to F$ is injective.* □

Once ϕ is known to be injective, we see that $F_1 \equiv \text{image}(\phi)$ is a copy of F with presentation $\langle x_1, x_2, \ldots | x_n^{x_i} = x_{n+1}$ for $1 \leqq i < n \rangle$. A glance at the defining presentation of F now yields:

Proposition 1.7. *F is the HNN extension of F_1 with respect to the monomorphism $(\phi | F_1)\colon F_1 \to F_1$, with x_0 as the stable letter.* □

Repeating with respect to $F_2 \equiv \text{image}(\phi^2)$, etc., we see that F is an infinitely iterated HNN extension.

Proposition 1.8. *F contains a free abelian subgroup of infinite rank.*

Proof. The elements $x_0 x_1^{-1}, x_2 x_3^{-1}, x_4 x_5^{-1}, \ldots,$ represented as homeomorphisms of $\mathbf{R}$, have disjoint supports. They therefore commute and are linearly independent. □

(1.9) **Finite presentation.** Two finite presentations of F have appeared in the literature. Both have two generators x_0 and x_1. Let x_n for $n \geqq 2$ be the word in x_0 and x_1 defined inductively by $x_n = x_{n-1}^{x_{n-2}}$. Then the two presentations are

$$\langle x_0, x_1 | x_2^{x_0} = x_3,\ x_3^{x_1} = x_4 \rangle \quad \text{and} \quad \langle x_0, x_1 | x_2^{x_0} = x_3,\ x_3^{x_0} = x_3^{x_1} \rangle.$$

The relations may also be written as $r_1 = r_2 = 1$ in the first presentation and $r_1 = r_3 = 1$ in the second, where $r_1 = [x_0 x_1^{-1}, x_2]$, $r_2 = [x_1 x_2^{-1}, x_3]$, and $r_3 = [x_0 x_1^{-1}, x_3]$. These three relators have lengths 10, 18, and 14 when written out in terms of x_0 and x_1.

It is not difficult to verify directly that the two presentations above define F. The first presentation will also come out of our work in §5 below, where we will exhibit a $K(F,1)$-complex whose 2-skeleton corresponds to that presentation.