**On the Baker's map and the Simplicity of the Higher Dimensional Thompson Groups $nV$**

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**Abstract**. *We show that the baker's map is a product of tranpositions (particularly pleasant involutions), and conclude from this that an existing very short proof of the simplicity of Thompson's group $V$ applies with equal brevity to the higher dimensional Thompson groups $nV$.*

Of the original groups $F \subseteq T \subseteq V$ of Richard Thompson (see [5]), all are infinite and finitely presented, and the last two are simple. An infinite family of groups $nV$, $n \in \{1, 2, 3, \ldots, \omega\}$, of which $1V = V$, is introduced in [2] where it is shown that $2V$ is infinite and simple and not isomorphic to $V$. A finite presentation for $2V$ is given in [3], it is shown that $nV$ and $mV$ are isomorphic only when $m = n$ in [1], and metric properties of $2V$ are studied in [4].

A very short argument that $V = 1V$ is generated by transpositions is given in Section 12 of [2], followed by an equally short argument based on this fact (due to Rubin) that $V$ is simple. It is also shown in that section that the baker's map (an element of $2V$) prevents the first argument from showing that $2V$ is generated by transpositions. As a result, the proof in [2] of the simplicity of $2V$ is rather involved and is based on calculations which show that the abelianization of $2V$ is trivial.

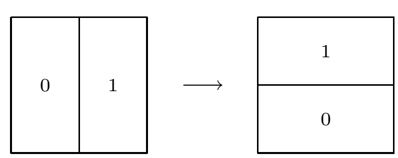
Here we give a short proof that the baker's maps in $2V$ are products of transpositions in $2V$. From there it is an easy exercise to combine this with the material in Section 12 of [2] to give a short proof of the simplicity of $2V$ and also to extend all the results to all of the $nV$, $n \leq \omega$.

Longer arguments for simplicity exist. Presentations (finite when $n < \omega$) for the $nV$, $n \leq \omega$, are given in [6], and one can calculate from these presentations that each of the $nV$, $n \leq \omega$, has trivial abelianization. From the arguments in Section 3 of [2] (which are about $2V$ but, as noted in 4.1 of [2], they apply as well to the $nV$, $n \leq \omega$) it then follows that each of the $nV$, $n \leq \omega$, is simple.

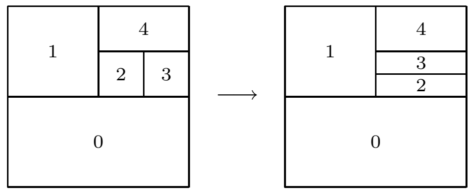
To keep this paper brief, we use notation, terminology and graphics from [2], and from this point we assume that the reader is familiar with their meanings.

**Lemma 1.** *Any baker's map in $2V$ is a product of finitely many proper transpositions from $2V$.*

The (primary) baker's map is given by the following.



A (secondary) baker's map is given by a pair of patterns that are identical and identically numbered with one exception: for one singly divided rectangle in the domain and for the corresponding singly divided rectangle in the range, the division is vertical in the domain and horizontal in the range. An example is below.

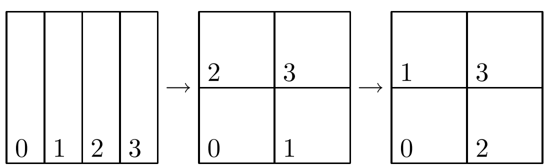



We refer to the rectangle containing the non-identity part of the baker's map as the *support* of the baker's map.

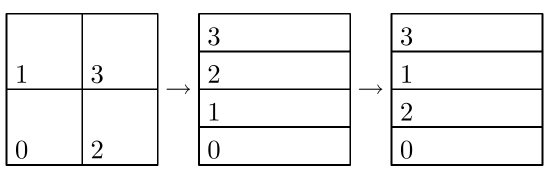
A transposition is given by a pair of identical patterns that are numbered identically except for a switch of two of the numbers. The transposition is *proper* if there are more than two rectangles in each pattern.

In the following, we say that two elements are identical modulo transpositions if each is a product of elements and the two products are identical if the transpositions are removed.

1. *Any baker's map $\beta$ is, modulo transpositions, a product of a baker's map whose support is the left half of the support of $\beta$, with a baker's map whose support is the right half of the support of $\beta$.* In the pictures below, we show only the support of $\beta$. The first arrow is the baker's map $\beta$ in a reducible form. The second is a proper transposition. The composition is the promised product of two baker's maps.



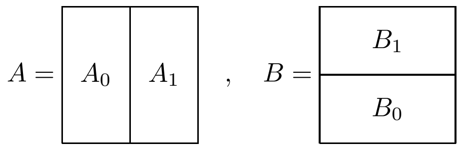

2. *Any baker's map $\beta$ is, modulo transpositions, a product of a baker's map whose support is the upper half of the support of $\beta$, with a baker's map whose support is the lower half of the support of $\beta$.* The relevant pictures follow and the comments are as in 1.



In the following, "arbitrarily small" means having support with diameter smaller than an arbitrarily chosen positive real.

3. *Any baker's map is, modulo transpositions, a product of arbitrarily small baker's maps.* This follows from 1 and 2.

4. *A product of a baker's map and an inverse of a baker's map with disjoint supports is a product of transpositions.* Let $A$ and $B$ be the disjoint supports. We refer to the rectangles in figures below to describe a sequence of transpositions. (a) Switch $A_0$ with $B_0$. (b) Switch $A_1$ with $B_1$. (c) Switch $A$ with $B$. The composition of (a) with (b) with (c) in that order is the desired result.




5. *If $R$ is a rectangle in a pattern so that neither side of $R$ has length more than $\frac{1}{2}$, then the baker's map with support $R$ is a product of transpositions.* The assumptions make $R$ one half of a rectangle $A$ that is not all of the unit square. Thus there is a rectangle $B$ that is disjoint from $A$. Let $S$ be the rectangle that is the "other half" of $A$. Let $\alpha$ be a product of a baker's map on $A$ with an inverse of a baker's map on $B$. By 4, this is a product of transpositions. By 1 or 2 we can modify $\alpha$ so that it is still a product of transpositions, and is a baker's map on each of $R$ and $S$ and an inverse of a baker's map on $B$. Let $\beta$ be a product of a baker's map on $B$ and an inverse of a baker's map on $S$. By 4 this is a product of transpositions. Now the composition of $\alpha$ with $\beta$ gives the desired result.

The lemma follows from 3 and 5. As discussed, this implies the following.

**Theorem 1.** *The $nV$, $n \leq \omega$, are generated by transpositions and are simple.*

## References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