Wind-up:

Using the facts that ~~an extract of a line is the~~, for any level of extraction (direct extracts, the direct extracts of the preceding ones, etc.), the extracts of a line determine the same function as the line, it follows [using the notion of partial table] that by replacing lines in a weak table (for a permutation) by a set of extracts, we can always arrange from two weak tables in 'nice' form (right sequence of A-lines ending in 1) to obtain the composition; Hence we have a way of representing the elements of G by finite (modulo A) artifacts — and multiplying them effectively. To check whether such an artifact represents the identity is easy: so G has a solvable W.P. — if A does.

Start.: Suppose that $A' = A$ (A': commutator subgroup of A).

To show that G is simple: say $a \in A$, $a \sim 1$ in the factor group, but $a \neq 1$. Then for some finite binary sequences σ, τ, we have $\sigma \neq \tau$ — in fact, σ and τ determine disjoint intervals — and $\sigma \to \tau$ is an extract of a line of A. Hence, if $X \in \mathcal{C}$, ~~and~~ $X \neq 1$, X_σ, X_τ are the 'similar' actions, of X, on σ and τ, we have $a^{-1} X_\sigma a = X_\tau$. Hence, $\underset{X_\tau =}{a^{-1} X_\sigma a} \sim X_\sigma$; so $X_\sigma \sim X_\tau$. As $X_\sigma \neq X_\tau$ (since $\sigma \neq \tau$ and $X \neq 1$), we obtain a relation in $\mathcal{C}'$. As $\mathcal{C}'$ is simple, this means $X \sim 1$ for all $X \in \mathcal{C}'$ in the factor group.

This easily collapses the whole group (for $g, h \in A^*$, $\exists\, p \in X$, $[(p^{-1} g p), h] = 1$ — hence $gh \sim hg$ & $\therefore$ A^* collapses to Abelian. But $A^* \cong A$, & $A' = A$, so A^* collapses too).

(Now apply the 2nd Postcard to get from A, a f.g. $\hat{A}$ (if A f.g.) s.t. $(\hat{A})' = \hat{A}$).

Let A, with a solvable word problem, be a group of permutations of the set $\mathcal{J}$ of positive integers and such that, for $j, k \in \mathcal{J}$ and $g \in A$, it is decidable whether or not $g(j) = k$. (Convention: $(gh)(j) = h(g(j))$).

Now let $\mathcal{F}$ be the set of all finite sets $\mathcal{B}$ of ordered pairs $\langle m, g\rangle$, $m \in \mathcal{J}$, $g \in A$, such that for all $m \in \mathcal{J}$, $g, h \in A$, $\langle m,g\rangle, \langle m,h\rangle \in \mathcal{B}$ implies $g = h$.

And let $h(\mathcal{B})$, for $\mathcal{B} \in \mathcal{F}$, $= \{\langle m+1, g\rangle : \langle m, g\rangle \in \mathcal{B}\}$. Then $h(\mathcal{B}) \in \mathcal{F}$.

Let us call an ordered pair $\langle \mathcal{B}, n\rangle$ where $\mathcal{B} \in \mathcal{F}$ and $n = 0$ or $n \in \mathcal{J}$ acceptable if for every $\langle m, g\rangle \in \mathcal{B}$ we have, for all $x \geq m$, $g(x) + n \geq m$.

For $n \in \mathcal{J} \cup \{0\}$, and $\mathcal{B}, \mathcal{C} \in \mathcal{F}$, let $\mathcal{B} \overset{*}{\underset{(n)}{}} \mathcal{C} = \{\langle m,g\rangle : (\langle m,g\rangle \in \mathcal{B} \;\&\; \forall h \in A.\ (\langle m-n, h\rangle \notin \mathcal{C})) \vee (\langle m-n, g\rangle \in \mathcal{C} \;\&\; \forall h \in A\,(\langle m,h\rangle \notin \mathcal{B})) \vee (\langle m, f\rangle \in \mathcal{B} \;\&\; \langle m-n, h\rangle \in \mathcal{C} \;\&\; g = fh)\}$. Then $\mathcal{B} \overset{*}{\underset{(n)}{}} \mathcal{C} \in \mathcal{F}$.

Let $\mathcal{B}[g]$ be $\{\langle 1, g\rangle\}$, for $g \in A$. Then $\mathcal{B}[g] \in \mathcal{F}$.

Regular lines are of the form: $\sigma \to \tau$ where σ and τ are finite binary sequences; σ is the left sequence, τ the right sequence of the line.
A-lines are of the form: $\sigma \xrightarrow{\mathcal{B}} \tau, n$ where σ and τ are finite binary sequences, $n \in \mathcal{J} \cup \{0\}$, and $\mathcal{B} \in \mathcal{F}$; again, σ is the left sequence, τ the right sequence of the line.

For σ a finite binary sequence, let $[\sigma] = \{\beta \in \mathcal{N} : \sigma \subset \beta\}$, where $\mathcal{N}$ is equal to the set of all binary sequences of length ω which are not eventually either 0 or 1.

The composition of lines: We define:

(1) $(\sigma \to \tau) \cdot (\tau \to \rho) = (\sigma \to \rho)$
(2) $(\sigma \to \tau) \cdot (\tau \xrightarrow{\mathcal{B}} \rho, n) = (\sigma \xrightarrow{\mathcal{B}} \rho, n)$
(3) $(\sigma \xrightarrow{\mathcal{B}} \tau, n) \cdot (\tau \to \rho) = (\sigma \xrightarrow{\mathcal{B}} \rho, n)$
(4) $(\sigma \xrightarrow{\mathcal{B}} \tau, n) \cdot (\tau \xrightarrow{\mathcal{C}} \rho, w) = (\sigma \xrightarrow{\mathcal{B} \overset{*}{\underset{(n)}{}} \mathcal{C}} \rho, n+w)$.

A ~~table~~ weak table Π is a finite collection of lines, all of whose left sequences are distinct, such that the set ξ of left sequences of the lines is such that $\{[\sigma] : \sigma \in \xi\}$ forms a partition of $\mathcal{K}$, and such that if $\sigma \xrightarrow{\mathcal{B}} \tau, m$ belongs to Π, then $\langle \mathcal{B}, n \rangle$ is acceptable.

The <u>direct extracts</u> of a regular line $\sigma \to \tau$ are the lines $\sigma^\frown 0 \to \tau^\frown 0$ and $\sigma^\frown 1 \to \tau^\frown 1$; The <u>direct extracts</u> of an A-line $\sigma \xrightarrow{\mathcal{B}} \tau, n$, <u>where</u> $\langle \mathcal{B}, n \rangle$ <u>is</u> <u>acceptable</u>, are the lines (where $m^* = \max\{m : \text{for some } g \in A,\ \langle m, g \rangle \in \mathcal{B}\}$):

$$\sigma^\frown 1 \xrightarrow{h(\mathcal{B})} \tau, n+1, \quad \sigma^\frown \langle 0 \rangle^{m^*+1} \to \tau^\frown \langle 1 \rangle^{m}{}^\frown \langle 0 \rangle^{m^*+1}, \text{ and, for } 0 < m \le m^*,$$

$$\begin{cases} \sigma^\frown \langle 0 \rangle^{m\frown} 1 \to \tau^\frown \langle 1 \rangle^{(g(m)+m)-m}{}^\frown \langle 0 \rangle^{g(m)}{}^\frown 1 & \text{provided that } \langle m, g \rangle \in \mathcal{B}, \\ \sigma^\frown \langle 0 \rangle^{m\frown} 1 \to \tau^\frown \langle 1 \rangle^{m}{}^\frown \langle 0 \rangle^{m}{}^\frown 1 & \text{otherwise (i.e., if for all } h \in A,\ \langle m, h \rangle \notin \mathcal{B}). \end{cases}$$

An <u>extract</u> of a line is a line obtained from it as the last in a series of direct extracts, starting from the given line.

<u>Lemma A</u>: If σ is a finite binary sequence terminating in 1, and ~~finite binary~~ σ extends σ', then if $\sigma' \xrightarrow{\mathcal{B}} \tau, n$ is an A-line and $\langle \mathcal{B}, n \rangle$ is acceptable, there is some line which is an extract of this line whose left sequence is σ; moreover, there is only one such line.

Given two weak tables, such that ~~the~~ every right sequence of the first, Π_1, ends in 1 if it belongs to an <u>A-line</u>, and is an extension of a left sequence of the second, Π_2, we define the <u>composition</u>, $\Pi_1 \circ \Pi_2$ as follows: We replace each line of Π_1 as follows: ~~the following line~~: The line $\sigma \to \tau$ or $\sigma \xrightarrow{\mathcal{B}} \tau, n$ is replaced, where τ' is the unique (by the partition property of the left sequences of the lines of Π_2) left sequence which τ extends which occurs in a line (likewise unique) of Π_2, and $\tau' \to \rho^{\Sigma}$ belongs to Π_2, by the composition of the Π_1 line with $\tau'^\frown \mu \to \rho^\frown \mu$, μ being the binary sequence such that $\tau = \tau'^\frown \mu$. If instead $\tau' \xrightarrow{\mathcal{C}} \rho, w$ is the unique line of Π_2 with τ' as its left sequence, the Π_1 line is replaced by the composition of the Π_1 line with the unique line, guaranteed by <u>Lemma A</u>, which is an extract of $\tau' \xrightarrow{\mathcal{C}} \rho, w$ and has τ as its left sequence (as τ extends τ' and τ ends in 1).

Every $\beta \in \mathcal{K}$ has the form $\langle 1\rangle^{i} \frown \langle 0\rangle^{j} \frown 1 \frown \beta'$ for some unique $\left\{\begin{matrix} j \in \mathcal{J} \\ i \in \mathcal{J}\cup\{0\}\end{matrix}\right\}$ and $\beta' \in$
$i-j$ is the parity of β, and j is its index.

The function defined by the line $\sigma \to \tau$ is that function χ with domain $[\sigma]$ such that $\chi(\sigma \frown \beta) = \tau \frown \beta$ for all $\beta \in \mathcal{K}$.

The function defined by the line $\sigma \xrightarrow{\mathfrak{B}} \tau, m$, where $\langle \mathfrak{B}, m\rangle$ is acceptable is that function χ with domain $[\sigma]$ such that:

$\{$ If β has parity $= -m$, index $= k$, and $\langle m, g\rangle \in \mathfrak{B}$,
Then $\chi(\sigma \frown \beta) = \tau \frown \langle 1\rangle^{(g(k)+m)-m} \frown \langle 0\rangle^{g(k)} \frown 1 \frown \beta'$.

(here $\beta = \langle 1\rangle^{k-m} \frown \langle 0\rangle^{k} \frown \beta'$ and $k \geq m$ — hence $g(k)+m \geq m$, $\langle \mathfrak{B}, m\rangle$ being accept

$\{$ If β does not have parity $= -m$ for any $\langle m, g\rangle \in \mathfrak{B}$,
then $\chi(\sigma \frown \beta) = \tau \frown \langle 1\rangle^{m} \frown \beta$.

The function defined by a weak table Γ is that function χ with domain $\mathcal{K}$ (as the left sequences of the lines of Γ yield intervals forming a partition of $\mathcal{K}$) which is the union of the functions defined by the lines.

A partial table is a set of regular lines and A-lines $\sigma \xrightarrow{\mathfrak{B}} \tau, m$ (such that $\langle \mathfrak{B}, m\rangle$ is acceptable) such that all the left sequences are distinct and incompatible; the function defined by the partial table is the union of the line function

For $g \in A$ let $g^* \in {}^{\mathcal{K}}\mathcal{K}$ be the function defined as follows:

$$\begin{cases} g^*(\langle 1\rangle^{m} \frown \langle 0\rangle^{m} \frown 1 \frown \beta) = \langle 1\rangle^{g(m)} \frown \langle 0\rangle^{g(m)} \frown 1 \frown \beta & \text{for } m \in \mathcal{J},\ \beta \in \mathcal{K} \\ g^*(\beta) = \beta & \text{otherwise.} \end{cases}$$

; then g^* is a permutation since g is.

Where $A^* = \{g^* : g \in A\}$, and $\mathcal{C}'$ is the group of permutations of $\mathcal{K}$ which are defined by weak tables with only regular lines (thus, to give permutations, it is necessary and sufficient that the right sequences also yield intervals which partition $\mathcal{K}$, the right sequences all being distinct), let G be the group generated by $A^* \cup \mathcal{C}'$. ($\mathcal{C}'$ is finitely generated)

Let G_0 be the group of all permutations p of $\mathcal{K}$ such that, for every $\beta \in \mathcal{K}$, there is some finite sequence σ such that $\beta = \sigma \frown \beta'$ and $p(\sigma \frown \gamma) = \tau \frown \gamma$ for some finite sequence τ, for all $\gamma \in \mathcal{K}$.