

Injectors in Finite Solvable Groups

Very preliminary report

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All groups considered here are finite and solvable. A *Sylow p -subgroup* of G is a subgroup whose order is a power of p and whose index is not divisible by p . A *Fitting set* of G is a set of subgroups of G that is closed under conjugation, the taking of (sub)normal subgroups, and the taking of products of mutually normalizing subgroups. An example is the set of subgroups of G that are of p -power order. If \mathcal{F} is a Fitting set of G , a subgroup H is an \mathcal{F} -*injector* of G if for each subnormal subgroup X of G , $H \cap X$ is maximal in X with respect to being contained in \mathcal{F} . Thus when \mathcal{F} is the set of p -subgroups of G , the \mathcal{F} -injectors of G are its Sylow p -subgroups, which form a conjugacy class of G . In general, the \mathcal{F} -injectors of G form a conjugacy class of G .

If there exists some Fitting set \mathcal{F} of G such that H is an \mathcal{F} -injector of G , we say H is an *injector* of G and write $H \in \text{inj}(G)$. All normal subgroups and all maximal subgroups of a finite solvable group G are injectors.

Note: If H is an injector of a finite solvable group G , then

(**) If X is subnormal G , then $H \cap X$ is pronormal in $N_G(X)$. [A is pronormal in B means A and A^b are conjugate in $\langle A, A^b \rangle$ for all $b \in B$.]

However, a subgroup satisfying (**) need not be an injector [See Exercise 2, p.553 in Doerk and Hawkes' *Finite Soluble Groups*, for example.]

A *2-maximal* subgroup of G is a non-maximal subgroup H such that $H < K < G$ implies K is maximal in G .

Claim. If H satisfies (**) in G and has a normal complement in G , then $H \in \text{inj}(G)$. Also, if H is 2-maximal in G and satisfies (**) in G , then $H \in \text{inj}(G)$.

The latter is a special case of the following: Let H satisfy property $(**)_{\mathcal{P}}$ if whenever $H \leq L \leq G$, H satisfies property (**) in L . Note: If $H \in \text{inj}(G)$ and $H \leq L \leq G$, then $H \in \text{inj}(L)$. This implies that if $H \in \text{inj}(G)$, then H satisfies $(**)_{\mathcal{P}}$ in G .

Claim. If H satisfies $(**)_{\mathcal{P}}$ in G , then $H \in \text{inj}(G)$.

A property nominally weaker than (**) is: (*) If X is normal in G then $H \cap X$ is normal in G .

We define $(*)_{\mathcal{P}}$ in a manner analogous to $(**)_{\mathcal{P}}$ to get a property nominally weaker than $(**)_{\mathcal{P}}$.

Brewster and Hauck have shown that $(*)_{\mathcal{P}}$ is equivalent to the *strong Frattini argument*.

Question. Is $(*)_P$ equivalent to $(**)_{P}$?

Claim. No. There exists a 2-maximal subgroup H of a finite solvable group G that does not satisfy $(**)$ but that does satisfy $(*)$, and, therefore, $(*)_P$; H is not an injector of G .