Here's a proof of $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$, that is something like I wanted you to do. (Of course, there are many other correct versions.)

PROOF by induction: Let P_n be the statement "$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$".

$n = 1$: The LHS of P_1 is

$$\begin{aligned}\sum_{k=1}^{1} k2^k &= 1 \cdot 2^1 = 2^1 \quad \text{(M3)} \\ &= 2 \quad (\text{Def of } 2^1),\end{aligned}$$

while the RHS of P_1 is

$$\begin{aligned}(1-1)2^{1+1} + 2 = (1 + (-1))2^{1+1} + 2 &= 0 \cdot 2^{1+1} + 2 \quad \text{(def. of subtraction and A4)} \\ &= 0 + 2 \quad (\text{since } \forall m \in \mathbb{Z},\ 0 \cdot m = 0) \\ &= 2 + 0 = 2 \quad \text{(Commutativity and A3)}\end{aligned}$$

Thus, P_1 holds.

Assuming P_n, we prove P_{n+1}. The LHS of P_{n+1} is

$$\begin{aligned}\sum_{k=1}^{n+1} k2^k &= \sum_{k=1}^{n} k2^k + (n+1)2^{n+1} \quad \text{(def. of summation)} \\
&= \Big((n-1)2^{n+1} + 2\Big) + (n+1)2^{n+1} \quad (\text{by } P_n) \\
&= (n+1)2^{n+1} + \Big((n-1)2^{n+1} + 2\Big) \quad \text{(Commutativity)} \\
&= \Big((n+1)2^{n+1} + (n-1)2^{n+1}\Big) + 2 \quad \text{(Associativity)} \\
&= \Big(2^{n+1}(n+1) + 2^{n+1}(n-1)\Big) + 2 \quad \text{(Commutativity)} \\
&= 2^{n+1}\Big((n+1) + (n + (-1))\Big) + 2 \quad \text{(D)} \\
&= 2^{n+1}\Big(n + \Big(1 + (n + (-1))\Big)\Big) + 2 \quad \text{(Associativity)} \\
&= 2^{n+1}\Big(n + \Big(1 + ((-1) + n)\Big)\Big) + 2 \quad \text{(Commutativity)} \\
&= 2^{n+1}\Big(n + \Big((1 + (-1)) + n\Big)\Big) + 2 \quad \text{(Associativity)} \\
&= 2^{n+1}\Big(n + \Big(0 + n\Big)\Big) + 2 \quad \text{(A4)} \\
&= 2^{n+1}\Big(n + \Big(n + 0\Big)\Big) + 2 \quad \text{(Commutativity)} \\
&= 2^{n+1}\Big(n + n\Big) + 2 \quad \text{(A3)} \\
&= 2^{n+1}\Big(n \cdot 1 + n \cdot 1\Big) + 2 \quad \text{(M3)} \\
&= 2^{n+1}\Big(n(1+1)\Big) + 2 \quad \text{(D)} \\
&= 2^{n+1}\Big(n \cdot 2\Big) + 2 \quad \text{(def. of 2)} \\
&= \Big(2^{n+1}n\Big)2 + 2 \quad \text{(Associativity)} \\
&= \Big(n2^{n+1}\Big)2 + 2 \quad \text{(Commutativity)} \\
&= n\Big(2^{n+1}2\Big) + 2 \quad \text{(Associativity)} \\
&= n2^{(n+1)+1} + 2 \quad \text{(def. of powers).}\end{aligned}$$

The RHS of P_{n+1} is

$$\begin{aligned}\Big((n+1) - 1\Big)2^{(n+1)+1} + 2 &= \Big((n+1) + (-1)\Big)2^{(n+1)+1} + 2 \\
&= \Big(n + (1 + (-1))\Big)2^{(n+1)+1} + 2 \quad \text{(Associativity)} \\
&= \Big(n + 0\Big)2^{(n+1)+1} + 2 \quad \text{(A4)} \\
&= n2^{(n+1)+1} + 2 \quad \text{(A3)}\end{aligned}$$

This proves P_{n+1}.