

HOMEWORK 4: DUE FRIDAY APRIL 17.

1) Prove exactly one of the following two:

a) Prove that if $n \in \mathbb{N}$ is composite (meaning not prime¹), then

$$n = ab,$$

where $a, b \in \mathbb{N}$ with $1 < a < n$ and $1 < b < n$.

b) Introduction to problem: Recall from class that a number of the form $M_n = 2^n - 1$ is called a **Mersenne number**. If M_n is prime, it's called a **Mersenne prime**. For instance, $M_2 = 2^2 - 1 = 3$ is prime, $M_3 = 2^3 - 1 = 7$ is prime, but $M_4 = 2^4 - 1 = 15$ is not prime. However, $M_5 = 2^5 - 1 = 31$ is prime again. It is not known if there exists infinitely many Mersenne primes.

Prove that if M_n is prime, then n is prime. You may *assume* Part a) of this problem. Suggestion: Prove the contrapositive! You will need the geometric series

$$x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + \cdots + x + 1) \quad (*)$$

2) Prove Proposition 6.14, page 43 of the book.

3) Let $a = a_n a_{n-1} \cdots a_1 a_0$, where $n \geq 0$ and $0 \leq a_i \leq 9$ for each i , be the base 10 (or decimal) representation of a natural number a . Recall that the notation $a = a_n a_{n-1} \cdots a_1 a_0$ means that

$$a = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0$$

Prove exactly one of the following three:

a) a is divisible by 6 if and only if a is even and $a_0 + a_1 + \cdots + a_n$ is divisible by 3.

b) a is divisible by 8 if and only if $a_2 a_1 a_0$ is divisible by 8.

c) a is divisible by 11 if and only if

$$(a_0 + a_2 + a_4 + \cdots) - (a_1 + a_3 + a_5 + \cdots) = \sum_{i=0}^n (-1)^i a_i$$

is divisible by 11.

Hint for c). Prove the following lemma: If $i \geq 0$ is even, then $10^i - 1$ is divisible by 11 and if $i \geq 1$ is odd, then $10^i + 1$ is divisible by 11. Subhint. For instance, to prove the first statement of the lemma, notice that if i is even, then $i = 2j$ for some j and you can write $10^i - 1$ as $100^j - 1$. Now use the "geometric sum formula" (*) above (with $x = 100$ in this case).

4) Prove exactly one of the following two:

a) Prove that if $m, n \in \mathbb{N}$, then $m \cdot n \in \mathbb{N}$ using the idea of a successor set as we studied in class.²

b) Prove that \mathbb{Z} is closed under subtraction and multiplication. Here, \mathbb{Z} is the subset of \mathbb{R} defined by $\mathbb{Z} = \{0\} \cup \mathbb{N} \cup (-\mathbb{N})$, where \mathbb{N} was constructed from \mathbb{R} using the idea of a successor set.

5) As we did in class, let \tilde{Z} be a set with operations of $\tilde{+}$ and $\tilde{\cdot}$ and let $\tilde{N} \subseteq \tilde{Z}$ be a subset, and suppose that \tilde{Z} and \tilde{N} satisfy the same axioms as \mathbb{Z} and \mathbb{N} . Let

$$f: \tilde{Z} \rightarrow \mathbb{Z}$$

be the function defined in class. Prove that for all $m, n \in \tilde{Z}$,

$$f(m \tilde{\cdot} n) = f(m) \cdot f(n) \quad (*).$$

Hint: First prove the **Fact**: For all $a \in \tilde{Z}$, $f(-a) = -f(a)$. Now *Fix* $m \in \tilde{N}$. Prove (*) for $n = \tilde{0}$. Then prove (*) for $n \in \tilde{N}$. Finally, prove (*) for $n \in -\tilde{N}$ using the **Fact**.

¹Recall that n is prime means that n is divisible by exactly two different natural numbers, 1 and n . In particular, 1 and n must be different for n to be prime.

²Recall that a successor set $A \subseteq \mathbb{R}$ is a set satisfying the following three properties: $0 \notin A$, $1 \in A$, and $n \in A \implies n + 1 \in A$. We defined $\mathbb{N} \subseteq \mathbb{R}$ as follows: An element $n \in \mathbb{R}$ belongs to \mathbb{N} if (and only if) n belongs to every successor set. We remarked, in particular, that \mathbb{N} is a subset of every successor set.