

TEST 2: DUE THURSDAY APRIL 30.

TEST RULES: You are allowed to use only your personal class notes, your personal previous HWs, the pdf files on our 330 website, and the 330 course pack. *Collaborating, the internet, other books, etc ... (basically any help whatsoever) are NOT allowed.*

1) In class I mentioned that there is no simple divisibility rule for dividing by 7 ... just in case you're wondering about it, here it is. It's not so easy to use but it works.

Example: Is the number 9,418,346 divisible by 7? Here is the trick: Look at the groups of three numbers divided by the commas and consider the alternating sum

$$346 - 418 + 9 = -63.$$

Since -63 is divisible by 7 (in fact, $-63 = 7 \cdot (-9)$) our test below says that 9,418,346 is divisible by 7.

Example: Is the number 12,100,944,814 divisible by 7? The trick is to look at the groups of three numbers divided by the commas and consider the alternating sum

$$814 - 944 + 100 - 12 = -42$$

Since -42 is divisible by 7 (in fact, $-42 = 7 \cdot (-6)$) our test below says that 12,100,944,814 is divisible by 7.

Here's the test.

Theorem: Let

$$a = a_n a_{n-1} \cdots a_5 a_4 a_3 a_2 a_1 a_0$$

be a natural number written in decimal (base 10) form. Then a is divisible by 7 if and only if the alternating sum of the successive groups of three digits:

$$\begin{aligned} a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - a_{11} a_{10} a_9 + \cdots &= \sum_{k=0,1,2,\dots} (-1)^k a_{3k+2} a_{3k+1} a_{3k} \\ &= \sum_{k \text{ even}} a_{3k+2} a_{3k+1} a_{3k} - \sum_{k \text{ odd}} a_{3k+2} a_{3k+1} a_{3k} \end{aligned}$$

is divisible by 7. Here, $a_{3k+2} a_{3k+1} a_{3k}$ is the decimal number $a_{3k+2} 10^2 + a_{3k+1} 10^1 + a_{3k}$. Here are some steps:

- i) Prove that $10^{3k} + 1$ where k is odd and $10^{3k} - 1$ where k is even are each divisible by 7.
- ii) Let $x := a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - a_{11} a_{10} a_9 + \cdots$ and prove that

$$a = x + 7b,$$

for some integer b . All you have to do is prove that $a - x$ is divisible by 7. To do so, begin by proving that

$$\begin{aligned} a &= a_n a_{n-1} \cdots a_5 a_4 a_3 a_2 a_1 a_0 \\ &= a_2 a_1 a_0 + a_5 a_4 a_3 10^3 + a_8 a_7 a_6 10^6 + \cdots \\ &= \sum_{k=0,1,2,\dots} a_{3k+2} a_{3k+1} a_{3k} 10^{3k} \end{aligned}$$

- iii) Now prove the theorem.
- iv) *Using the theorem*, verify BY HAND (not by a calculator!!!) whether or not the following numbers are divisible by 7:

$$57,092 \text{ and } 49,058.$$

- 2) (**Perfect numbers**) A number $n \in \mathbb{N}$ is said to be **perfect** if it is the sum of its proper divisors (divisors excluding itself). For example, $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are perfect. It's not known if there exists any odd perfect numbers! In this problem we prove the following theorem.

Theorem: If $n = 2^m(2^{m+1} - 1)$ where $m \in \mathbb{N}$ and $2^{m+1} - 1$ is prime, then n is perfect.

Proceed as follows:

- i) **Lemma:** Let $c, d \in \mathbb{N}$, $c, d > 1$, and write $c = p_1 \cdots p_\ell$ and $d = q_1 \cdots q_m$ as products of their prime factors. (We know that c and d have unique such factorizations according to the Fundamental Theorem of Arithmetic.) Let $a \in \mathbb{N}$, $a > 1$, and write $a = t_1 \cdots t_n$ as a product of its prime factors. Prove that if a divides $c \cdot d$, then each t_i equals one of the p_j 's or q_j 's. In other words,

$$a = t_1 \cdots t_n$$

where for each i , $t_i = p_j$ or $t_i = q_j$ for some j . (Hint: We know that $x := c \cdot d = a \cdot b$ for some natural number b . Write x as a product of prime numbers using the prime factors of c and d and then using the prime factors of a and b . Finally, use the uniqueness part of the Fundamental Theorem of Arithmetic.)

- ii) If $q = 2^{m+1} - 1$ where $m \in \mathbb{N}$ and q is prime, using the lemma prove that the proper divisors of $n = 2^m(2^{m+1} - 1) = 2^m q$ are $1, 2, \dots, 2^m, q, 2q, \dots, 2^{m-1}q$.
 iii) Now prove the theorem.
 iv) What does m equal for the perfect numbers $n = 6$ and $n = 28$?

- 3) Let \tilde{Z} be a set with operations of $\tilde{+}$ and $\tilde{\cdot}$ and let $\tilde{N} \subseteq \tilde{Z}$ be a subset, and suppose that \tilde{Z} and \tilde{N} satisfy the same axioms as \mathbb{Z} and \mathbb{N} . In particular,

$$\tilde{Z} = \{\tilde{0}\} \cup \tilde{N} \cup (-\tilde{N}),$$

where $\tilde{0}$ is the “zero” of \tilde{Z} . We proved in class (and in HW) that there is a bijection

$$f: \tilde{Z} \rightarrow \mathbb{Z}$$

such that for all $m, n \in \tilde{Z}$,

$$f(m \tilde{+} n) = f(m) + f(n) \quad \text{and} \quad f(m \tilde{\cdot} n) = f(m) \cdot f(n),$$

properties you are allowed to use in this problem. (Recall that by definition $f(\tilde{0}) := 0$, and f is defined on \tilde{N} using induction ... you can ask me how f is defined if you wish although you don't need all the information about how f is defined to do this problem.) Assume that $\tilde{Z} \subseteq \tilde{Q}$ where \tilde{Q} is a set of “rational numbers” having the same properties as our usual set of rational numbers. For example, for each $r \in \tilde{Q}$ with $r \neq \tilde{0}$ there is a number $r^{-1} \in \tilde{Q}$ such that

$$r \tilde{\cdot} r^{-1} = \tilde{1},$$

where $\tilde{1}$ is the multiplicative identity. Moreover,

$$\tilde{Q} = \left\{ m/n = m \tilde{\cdot} n^{-1} \mid m \in \tilde{Z}, n \in \tilde{N} \right\}.$$

If $a, b \in \tilde{Q}$ with $b \neq \tilde{0}$ we write (just as for the usual rational numbers) a/b for $a \tilde{\cdot} b^{-1}$.

- i) Let $a, m \in \tilde{Z}$ and $b, n \in \tilde{N}$. Prove that if $a/b = m/n$, then $f(a)/f(b) = f(m)/f(n)$.
 ii) Define $F: \tilde{Q} \rightarrow \mathbb{Q}$ as follows: If $x \in \tilde{Q}$, then $x = a/b$ for some $a \in \tilde{Z}$ and $b \in \tilde{N}$; we define¹

$$F(x) := f(a)/f(b).$$

Prove *either* that F is injective or F is onto (surjective).

- iii) Let $x \in \tilde{Q}$ and $y \in \tilde{Q}$. Prove one of the following two equalities:

$$F(x \tilde{+} y) = F(x) + F(y),$$

$$F(x \tilde{\cdot} y) = F(x) \cdot F(y).$$

¹If you want you (but you don't have to) give a short remark why F is well-defined.