# HOME WORK 3

## 1. Introduction to Lebesgue Measure

**Exercises 1.1.**

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a nondecreasing function; thus, if $x \leq y$, then $f(x) \leq f(y)$. Define the *set function* $f : \mathscr{I}_1 \longrightarrow [0, \infty)$ by $f(a, b] = f(b) - f(a)$. Prove that the set function $f$ is finitely additive. An additive set function of this sort is called a **Lebesgue-Stieltjes additive set function**. In particular, for $f = x$, we get the usual Lebesgue measure.
2. In this problem we look at examples of Lebesgue-Stieltjes set functions.
   (a) Given a nonnegative function $g : \mathbb{R} \longrightarrow \mathbb{R}$ that is Riemann integrable on any finite interval, we define $\mu_g : \mathscr{I}_1 \longrightarrow [0, \infty)$ by integration:
   $$\mu_g(a, b] = \int_a^b g(x)\, dx.$$
   In particular, if $g = 1$, this is just the usual Lebesgue measure. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an nondecreasing continuously differentiable function. Show that as set functions, $f = \mu_{f'}$.
   (b) Let $H : \mathbb{R} \longrightarrow \mathbb{R}$ be the 'Heaviside function':
   $$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$
   Describe what $H$ does to sets in $\mathscr{I}_1$. Do you see why as a set function, $H$ is called the **Dirac set function**? Recall that the so-called 'Dirac delta function' $\delta(x)$ is defined by the properties
   $$\int_a^b \delta(x)\, dx = \begin{cases} 1 & \text{if } (a, b) \text{ contains } 0 \\ 0 & \text{otherwise.} \end{cases}$$
   In view of Part (a), do you see why we formally write $H' = \delta$?
3. In this exercise we prove that Lebesgue-Stieltjes set functions are the only additive set functions on $\mathscr{I}_1$.
   (a) Let $\mu : \mathscr{I}_1 \longrightarrow [0, \infty)$ be an additive set function and define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by
   $$f(x) = \begin{cases} -\mu(x, 0] & \text{if } x < 0 \\ \mu(0, x] & \text{if } x \geq 0. \end{cases}$$
   Show that $f$ is nondecreasing and as a set function, $f = \mu$.
   (b) Show that if $f = g$ as set functions, then $f$ and $g$ differ by a constant. So, the function corresponding to a Lebesgue-Stieltjes set function is essentially unique.
4. In this exercise, we study the translation invariance of Lebesgue measure on $\mathscr{I}_1$. Related properties for $\mathbb{R}^n$ are studied later in the course. Given $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, we denote the translation of $A$ by $x$ by $A + x$ or $x + A$:
   $$x + A = A + x = \{a + x\,;\, a \in A\} = \{y \in \mathbb{R}\,;\, y - a \in A\}.$$

(a) Prove that $\mathscr{I}_1$ is translation invariant in the sense that if $I \in \mathscr{I}_1$, then $x + I \in \mathscr{I}_1$ for any $x \in \mathbb{R}$.

(b) A **translation invariant set function** $\mu : \mathscr{I}_1 \longrightarrow [0, \infty)$ is a set function satisfying $\mu(x + I) = \mu(I)$ for all $x \in \mathbb{R}$ and $I \in \mathscr{I}_1$. Prove that Lebesgue measure $\mathfrak{m} : \mathscr{I}_1 \longrightarrow [0, \infty)$ is translation invariant.

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a nondecreasing function. We prove that the Lebesgue-Stieltjes set function defined by $f$ is translation invariant if and only if $f$ is affine, that is, $f = ax + b$ for some $a, b \in \mathbb{R}$. Thus, up to an affine transformation, Lebesgue measure (which corresponds to $f = x$) is the unique translation invariant Lebesgue-Stieltjes set function on $\mathscr{I}_1$. Proceed as follows.
   (a) Show that if $f$ is affine, then the Lebesgue-Stieltjes set function defined by $f$ is translation invariant.
   (b) Henceforth, assume that the Lebesgue-Stieltjes set function defined by $f$ is translation invariant. Let $g(x) = f(x) - f(0)$. Show that $g$ is linear, that is, $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$.
   (c) Prove that $g(rx) = r\, g(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}$. In particular, $g(r) = a\, r$ where $a = g(1)$. Suggestion: First prove that $g(nx) = n\, g(x)$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.
   (d) Prove that $g(x) = a\, x$ for all $x \in \mathbb{R}$. From this, deduce that $f$ is affine. Suggestion: Use that $\mathbb{Q}$ is dense in $\mathbb{R}$ and that $g$ is nondecreasing.

## 2. Semirings, rings, $\sigma$-rings, $\sigma$-algebras, and monotone classes

**Exercises 2.1.**

1. Let $A \subset \mathbb{R}$ be any set and let $\mathscr{I}$ be the collection of all left-half open intervals with end points in $A$. Prove that $\mathscr{I}$ is a semiring.
2. Give an example of two semirings (say of the form described in the previous problem) whose intersection is not a semiring. Because of this poor structure, some authors do not work with semirings to do measure theory. However, even with this poor structure, it is possible to develop quite a bit of measure theory on semirings, as we shall demonstrate in class.
3. Here's an 'abstract' example of a semiring that is needed to prove the celebrated Riesz Representation Theorem, which we'll prove in 5-6 weeks. Let $\mathscr{F}$ be any collection of functions on a nonempty set such that if $f, g \in \mathscr{F}$, then $\max\{f, g\}$ and $\min\{f, g\}$ are also in $\mathscr{F}$. Show that the system of 'left-half open intervals' $\mathscr{I}_{\mathscr{F}}$, consisting of sets of the form
$$(f, g] = \{(x, t) \in X \times \mathbb{R}\, ;\, f(x) < t \leq g(x)\},$$
   where $f, g \in \mathscr{F}$ with $f \leq g$, is a semiring.
4. More examples of various classes.
   (a) Let $X$ be any set and let $A \subset X$ be any set. Let $\mathcal{A} = \{A\}$ be a single element collection of subsets of $X$. What are $\mathscr{R}(\mathcal{A})$, $\mathscr{R}_\sigma(\mathcal{A})$, $\mathscr{M}(\mathcal{A})$, and $\mathscr{S}(\mathcal{A})$ (the ring, $\sigma$-ring, monotone class, and $\sigma$-algebra, respectively, generated by $\mathcal{A}$)?
   (b) Let $X$ be any set and let $\mathscr{R}$ be the collection of subsets $A \subset X$ such that either $A$ is finite or $A^c$ is finite. Show that $\mathscr{R}$ is a ring but not necessarily a $\sigma$-ring.
   (c) Let $X$ be an uncountable set and let $\mathscr{S}$ be the collection of subsets $A \subset X$ such that either $A$ is countable or $A^c$ is countable. Show that $\mathscr{S}$ is a $\sigma$-algebra.

5. Let $\mathscr{R}$ be a ring of sets. For sets $A, B \in \mathscr{R}$, define 'addition' and 'multiplication' of the two sets by, respectively,

$$A \cdot B = A \cap B, \qquad A + B = (A \setminus B) \cup (B \setminus A).$$

(The right-hand side of $A + B$ is called the **symmetric difference** of $A$ and $B$ and is usually denoted by $A \Delta B$.) With these operations, prove that $\mathscr{R}$ is a ring in the algebraic sense of the word. Also, prove that $\mathscr{R}$ has the following properties:

$$A \cdot A = A, \quad A + A = 0 \qquad \text{for all } A \in \mathscr{R}.$$

Any ring with these properties is called a **Boolean ring**.

6. Let $f : X \longrightarrow Y$ be any map. If $\mathcal{A}$ is a ring, $\sigma$-ring, or monotone class of subsets of $Y$, prove that $f^{-1}(\mathcal{A})$ is a ring, $\sigma$-ring, or monotone class, respectively, of subsets of $X$. Also, prove that given any collection of subsets $\mathcal{A}$ in the range of $f$, we have $\mathscr{R}(f^{-1}(\mathcal{A})) = f^{-1}\mathscr{R}(\mathcal{A})$, $\mathscr{R}_\sigma(f^{-1}(\mathcal{A})) = f^{-1}\mathscr{R}_\sigma(\mathcal{A})$, and $\mathscr{M}(f^{-1}(\mathcal{A})) = f^{-1}(\mathscr{M}(\mathcal{A}))$.
7. Initiated by Dynkin in 1959, some authors introduce the notion of $\pi$-(or product) classes and $\lambda$-(or lattice) classes to study $\sigma$-algebras. A **$\pi$-class** is a collection of subsets of a set $X$ that is closed under finite intersections, while a **$\lambda$-class** is a monotone class containing the whole space $X$ having the property that if $A$ and $B$ are elements of the class with $B \subset A$, then $A \setminus B$ is in the class.
   (a) Show that an arbitrary intersection of $\pi$-classes or of $\lambda$-classes of subsets of $X$ is again a $\pi$-class or a $\lambda$-class.
   (b) Show that a collection of subsets of $X$ is a $\sigma$-algebra if and only if it is both a $\pi$-class and a $\lambda$-class.
   (c) Let $\mathscr{P}$ be a $\pi$-class. Show that $\mathscr{S}(\mathscr{P})$, the $\sigma$-algebra generated by $\mathscr{P}$, is the smallest $\lambda$-class containing $\mathscr{P}$.

## 3. Borel sets

**Exercises 3.1.**

1. Prove that any open set is a union of boxes in $\mathscr{I}_n$ whose end points are rational.
2. Prove that the Borel subsets of $\mathbb{R}^n$ is the $\sigma$-algebra generated by the collection of all (i) dyadic cubes; (ii) left-half open boxes with rational end points; (iii) left-half open boxes with dyadic end points; (iv) closed sets; (v) compact sets.
3. The technique used to prove the Dyadic cube theorem is useful in establishing analogous statements concerning open sets. For instance, imitating the proof of the Dyadic cube theorem, show that each open set in $\mathbb{R}^n$ is a countable union of closed cubes with disjoint interiors. In a similar way, prove that any open set in $\mathbb{R}$ is a countable disjoint union of open intervals. Is this last statement true in $\mathbb{R}^n$ for $n > 1$? Suggestion: For each $x$ in the open set in $\mathbb{R}$, there is a largest open interval containing $x$, say $I_x$. Prove that these intervals are disjoint and the open set is the union of all such intervals.
4. Given $x \in \mathbb{R}^n$, $r \in \mathbb{R}$, and $A \subset \mathbb{R}^n$, we denote the translation of $A$ by $x$ by $A + x$ or $x + A$, and the multiple of $A$ by $r$ by $rA$:

$$x + A = A + x = \{a + x\,;\, a \in A\}, \qquad rA = \{rx\,;\, x \in A\}.$$

Prove that $\mathscr{B}_n$ is translation and scalar invariant. That is, $\mathscr{B}_n + x = \mathscr{B}_n$ and $r\mathscr{B}_n = \mathscr{B}_n$ (for $r \neq 0$). Suggestion: Show that $\mathscr{I}_n$ is translation and scalar invariant first.

5. For $\mathbb{R}^n$, the $\sigma$-algebra generated by the open sets and by the compact sets are the same, namely the Borel sets. For a general topological space, they need not be the same. A subset in a topological space is said to be **$\sigma$-compact (or $\sigma$-bounded)** if it is contained in a countable union of compact sets.
   (a) Prove that for any $\sigma$-compact Hausdorff space, the Borel sets coincides with the $\sigma$-algebra generated by the compact sets. Suggestion: Recall that for a Hausdorff space, compact sets are closed.
   (b) Suppose that the topology of $X$ consists of just the sets $\varnothing$ and $X$. What are the Borel sets? What is the $\sigma$-algebra generated by the compact sets?

## 4. Integration of step functions on semirings

**Exercises 4.1.**

1. In this problem, we give an example connecting integration with sums. Let $\mathscr{P}(\mathbb{N})$ be the power set of the natural numbers. Consider the 'counting function' $\# : \mathscr{P}(\mathbb{N}) \longrightarrow [0,\infty]$ defined by $\#(A) =$ number of elements in $A$, if $A$ is a finite set, or $\#(A) = \infty$ if $A$ has infinitely many elements.
   (a) Show that $\#$ defines a measure on $\mathscr{P}(\mathbb{N})$.
   (b) Show that any $\mathscr{P}(\mathbb{N})$-simple function is of the form $f = \sum_{n=1}^N a_n\, \chi_n$, where $a_n \in \mathbb{R}$, and $\chi_n$ is defined by $\chi_n(x) = 0$ if $x \neq n$ and $\chi_n(x) = 1$ if $x = n$.
   (c) Given such a simple function, show that
   $$\int f\, d\# = \sum_{n=1}^N a_n.$$
2. In this exercise, we continue the discussion of Lebesgue-Stieltjes additive set functions started in Exercises 1.1.
   (a) Let $g$ be a nonnegative $\mathscr{I}_1$-step function. Prove that for any $\mathscr{I}_1$-step function $f$, we have
   $$\int f\, d\mu_g = \int f\, g\, dx.$$
   Recall that $\mu_g : \mathscr{I}_1 \longrightarrow [0,\infty)$ is defined by $\mu_g(a,b] = \int_a^b g(x)\, dx$. Notationally, do both sides of this equation agree if $g$ is the function 1?
   (b) Let $g$ be a continuously differentiable nondecreasing function on $\mathbb{R}$. Prove that for any $\mathscr{I}_1$-step function $f$, we have
   $$\int f\, dg = \int f\, g'\, dx.$$
   On the left-hand side, "$g$" is the set function $g : \mathscr{I}_1 \longrightarrow [0,\infty)$ defined by $g(a,b] = g(b) - g(a)$. Notationally, do both sides of this equation agree if $g$ is the function $x$?
3. Let $\mu : \mathscr{I} \longrightarrow [0,\infty]$ be an additive set function on a semiring $\mathscr{I}$ and let $g$ be a nonnegative $\mathscr{I}$-step function. Define $\mu_g : \mathscr{I} \longrightarrow [0,\infty]$ by
   $$\mu_g(A) = \int \chi_A\, g\, d\mu, \qquad \text{for all } A \in \mathscr{I}.$$
   (a) Prove that $\mu_g : \mathscr{I} \longrightarrow [0,\infty]$ is additive.
   (b) Prove that for any $\mathscr{I}_1$-step function $f$, we have
   $$\int f\, d\mu_g = \int f\, g\, d\mu.$$

4. Let $\mu : \mathscr{I} \longrightarrow [0,\infty]$ be map on a semiring $\mathscr{I}$ such that $\mu(A) < \infty$ for some $A$, and if $A \in \mathscr{I}$ with $A = \bigcup_{n=1}^N A_n$, $A_n \in \mathscr{I}$ disjoint, then
$$\mu(A) = \sum_{n=1}^N \mu(A_n).$$
Show that $\mu(\varnothing) = 0$. Thus, the requirement that $\mu(\varnothing) = 0$ in the definition of an additive set function is redundant if $\mu$ is not identically $\infty$.
5. Show that
$$\chi_{A\cap B} = \chi_A \cdot \chi_B,$$
$$\chi_{A^c} = 1 - \chi_A$$
$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B.$$
6. We give Bourbaki's proof of Problem 5 in Exercises 2.1. Let $\mathscr{R}$ be a ring of subsets of a set $X$, and let Let $\mathbb{Z}_2^X$ be the ring of $\mathbb{Z}_2$-valued functions on $X$. Show that as elements of $\mathbb{Z}_2^X$, we have
$$\chi_{A\Delta B} = \chi_A + \chi_B,$$
where $A\Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of $A$ and $B$. Now show that $\mathscr{R}$, with its operations of multiplication and additive given by intersection and symmetric differences, respectively, is isomorphic to a subring of $\mathbb{Z}_2^X$. Finally, show that $\mathscr{R}$ is isomorphic to the whole of $\mathbb{Z}_2^X$ if and only if $\mathscr{R}$ is the power set of $X$.

## 5. Properties of finitely additive set functions on semirings and rings

**Exercises 5.1.**

1. Prove that $\mathfrak{m} : \mathscr{I}_1 \longrightarrow [0,\infty]$ is a measure directly as follows. Let $(a,b] = \bigcup_{n=1}^\infty (a_n, b_n]$ be a countable union of disjoint sets. We may assume that none of the intervals is empty. Show that the countable union of $(a,b]$ can be rewritten as
$$(a,b] = \bigcup_{n=1}^\infty (c_n, c_{n-1}]$$
where $b = c_0 > c_1 > \cdots > c_n \to a$ as $n \to \infty$. Using this decomposition of $(a,b]$, prove that $\mathfrak{m}$ is countably additive on $\mathscr{I}_1$.
2. In this exercise, we give sufficient and necessary conditions so that Lebesgue-Stieltjes set functions are measures.
   (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the nondecreasing function
   $$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0. \end{cases}$$
   Show that $f : \mathscr{I}_1 \longrightarrow [0,\infty)$ is *not* a measure, that is, find a set on which $f$ is not countably additive. Note that $f$ is not right continuous.
   (b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be nondecreasing. Prove that $f : \mathscr{I}_1 \longrightarrow [0,\infty)$ is a measure if and only if $f$ is right continuous, that is, $f(x) = f(x+)$ at each point $x \in \mathbb{R}$. Suggestion: For necessity, given any $x$, let $x_1 > x_2 > x_3 > \cdots$ be any sequence decreasing to $x$, and consider the disjoint union $(x, x_1] = \bigcup_{n=1}^\infty (x_{n+1}, x_n]$.

3. For any nonnegative function $g : \mathbb{R} \longrightarrow \mathbb{R}$ that is Riemann integrable on any finite interval, prove that $\mu_g : \mathscr{I}_1 \longrightarrow [0, \infty)$, defined by $\mu_g(a, b] = \int_a^b g(x)\,dx$, is a measure.
4. The notion of $\sigma$-finite will occur quite often in future chapters. An additive set function $\mu$ on a semiring $\mathscr{I}$ of subsets of a set $X$ is said to be $\sigma$-finite if $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}$ is a sequence of disjoint sets in $\mathscr{I}$ with $\mu(X_n) < \infty$ for each $n$. Most measures of practical interest are $\sigma$-finite.
   (a) Show that Lebesgue measure on $\mathscr{I}_n$ and any Lebesgue-Stieltjes set function on $\mathscr{I}_1$ are $\sigma$-finite.
   (b) Prove that $\mu$ is $\sigma$-finite if $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}$ is a sequence of not necessarily disjoint sets in $\mathscr{I}$ with $\mu(X_n) < \infty$ for each $n$. Suggestion: Use the Fundamental Lemma of Semirings.
5. Let $\mu : \mathscr{R} \longrightarrow [0, \infty]$ be an additive function on a ring $\mathscr{R}$. Given sets $A$ and $B$ in $\mathscr{R}$, prove that
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$
   Warning: Be careful that at each step of your argument you never subtract two quantities because you could have a nonsense statement like $\infty - \infty$.
6. Let $\mu : \mathscr{R} \longrightarrow [0, \infty]$ be an additive function on a ring $\mathscr{R}$. Given sets $A, B, C$ in $\mathscr{R}$, prove that
$$\mu(A \cup B \cup C) + \mu(A \cap B) + \mu(B \cap C) + \mu(A \cap C) = \mu(A) + \mu(B) + \mu(C) + \mu(A \cap B \cap C).$$
7. Generalize the previous exercises as follows. Let $\mu : \mathscr{R} \longrightarrow [0, \infty)$ be an additive function on a ring $\mathscr{R}$. Given sets $A_1, \dots, A_N$ in $\mathscr{R}$, establish the **Poincaré formula**:
$$\mu\Big(\bigcup_{n=1}^{N} A_n\Big) = \sum_{n=1}^{N} \mu(A_n) - \sum_{1 \le i < j \le N} \mu(A_i \cap A_j) + \sum_{1 \le i < j < k \le N} \mu(A_i \cap A_j \cap A_k) - \cdots + (-1)^{N-1} \mu\Big(\bigcap_{n=1}^{N} A_n\Big).$$
   Taking all the negative terms to the left, show that the resulting formula holds even if $\mu : \mathscr{R} \longrightarrow [0, \infty]$. Suggestion: Use induction.