HOME WORK 5

1. Measurable functions

Exercises 1.1.

1. (a) Prove that a nonnegative function f is measurable if and only if all sets of the form $f^{-1}(k/2^n, (k+1)/2^n]$, where $k, n \in \mathbb{N}$ with $0 \le k \le 2^{2n} - 1$, and $f^{-1}(2^n, \infty]$, are measurable.
   (b) Prove that an extended real-valued function f is measurable if $f^{-1}(\pm\infty)$ and all sets of the form $f^{-1}(k/2^n, (k+1)/2^n]$, where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, are measurable. In fact, if $\{a_n\}$ is any countable dense subset of $\mathbb{R}$, prove that f is measurable if $f^{-1}(\pm\infty)$ and all sets of the form $f^{-1}(a_m, a_n]$ are measurable.
2. Following Halmos, we show that the composition of two Lebesgue measurable function is not necessarily Lebesgue measurable. Let φ and M be the homeomorphism and Lebesgue measurable set, respectively, of the previous homework. Let $g = \chi_M$. Show that $g \circ \varphi^{-1}$ is not Lebesgue measurable. Note that both φ^{-1} and g are Lebesgue measurable.
3. In the following series of problems, we study various convergence properties of measurable functions. We shall work with a fixed measure space $(X, \mathscr{S}, \mu)$. A measurable function f is **a.e. real-valued** if the set $\{x\,;\, f(x) = \pm\infty\}$ has measure zero. Let f_n, $n = 1, 2, \ldots$, and f be a.e. real-valued, measurable functions. Prove that $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$,

$$\mu\Big(\bigcap_{n=1}^{\infty} \bigcup_{m \ge n} \{x\,;\, |f_m(x) - f(x)| \ge \varepsilon\}\Big) = 0.$$

   Of course, the difference $f_m(x) - f(x)$ is not defined at points where an expression of the form $\pm\infty - \pm\infty$ results; the convention is to set this expression equal to zero at such points. The set of such points has measure zero, so is 'negligible'. Thus, in proofs, we usually assume that f and each f_n are real-valued.
4. We now study the notion of almost uniform convergence. A sequence $\{f_n\}$ of a.e. real-valued, measurable functions is said to **converge almost uniformly** to a measurable function f, denoted by $f_n \to f$ a.u., if for each $\varepsilon > 0$, there exists a measurable set A such that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly on $A^c = X \setminus A$ as $n \to \infty$. Recall that the last statement means that given any $\eta > 0$,

$$|f_n(x) - f(x)| < \eta, \quad \text{for all } x \in A^c \text{ and } n \text{ sufficiently large.}$$

   Note that f is necessarily a.e. real-valued.
   (a) Let $X = \mathbb{R}$ with Borel measure and let f_n be the characteristic function of the interval $[n, \infty)$. Show that $f_n \to 0$ pointwise, but $f_n \not\to 0$ a.u.
   (b) Let $X = [0, 1]$ with Borel measure and let f_n be the characteristic function of the interval $[1 - 1/n, 1]$. Show that $f_n \to 0$ a.u., but $f_n \not\to 0$ uniformly. Thus, a.u. convergence and uniform convergence are distinct notions.

The key difference between (a) and (b) is that in (b), the total space has finite measure. Egoroff's theorem (see Problem 7) states that in a finite measure space, pointwise convergence always implies a.u. convergence.

5. We give a characterization of almost uniform convergence. Let f_n and f be a.e. real-valued, measurable functions. Prove that $f_n \to f$ a.u. if and only if for each $\varepsilon > 0$,
$$\lim_{n\to\infty} \mu\Big(\bigcup_{m\geq n} \{x\,;\, |f_m(x) - f(x)| \geq \varepsilon\}\Big) = 0.$$
We may assume that f and each f_n are real-valued. You may proceed as follows.
   (a) Necessity: Fix $\varepsilon > 0$ and let $A_m = A_m(\varepsilon) = \{x\,;\, |f_m(x) - f(x)| \geq \varepsilon\}$. Then $A_m^c = \{x\,;\, |f_m(x) - f(x)| < \varepsilon\}$. Let $\eta > 0$. Show that there is a measurable set A with $\mu(A) < \eta$ and an integer N such that $A^c \subset \bigcap_{m\geq N} A_m^c$, that is, $\bigcup_{m\geq N} A_m \subset A$. Conclude the proof of necessity.
   (b) Sufficiency: Fix $\varepsilon > 0$. By hypothesis, for each n, there is an $N(n)$ such that $\mu(\bigcup_{m\geq N(n)} A_m(1/n)) < \varepsilon/2^n$. Let $A = \bigcup_{n=1}^{\infty} \bigcup_{m\geq N(n)} A_m(1/n)$. Show that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly on A^c.
6. Using Problems 3 and 5, prove that if a sequence $\{f_n\}$ of a.e. real-valued, measurable functions converges a.u. to a measurable function f, then the sequence converges to f a.e.
7. Egoroff's theorem gives the converse to the previous problem, provided that X has finite measure.

   Theorem 1.2 (Egoroff's Theorem)**.** *Suppose that X has finite measure. If a sequence $\{f_n\}$ of a.e. real-valued, measurable functions converges a.e. to an a.e. real-valued, measurable function f, then $\{f_n\}$ converges to f almost uniformly.*

   Suggestion: Use Problem 3 and 5.
8. We now prove that any Lebesgue measurable function is 'nearly continuous'.

   Theorem 1.3 (Lusin's Theorem)**.** *Let f be an a.e. real-valued, Lebesgue measurable function on a measurable set $A \subset \mathbb{R}^n$ of finite measure. Then given any $\varepsilon > 0$, there exists a closed set $C \subset A$ such that $\mathfrak{m}(A \setminus C) < \varepsilon$ and the restriction of f to C is a continuous function.*

   You may proceed as follows.
   (a) First prove the theorem for simple functions. Thus, let $f = \sum_{k=1}^{N} a_k \chi_{A_k}$ be a simple function supported on A with $A = \bigcup_{k=1}^{N} A_k$ and the A_k are disjoint measurable sets. Suggestion: Given $\varepsilon > 0$, produce a closed set $C_k \subset A_k$ with $\mathfrak{m}(A_k \setminus C_k) < \varepsilon/N$. (Use a previous homework problem). Let $C = \bigcup_{k=1}^{N} C_k$.
   (b) Since any measurable function is the difference of its nonnegative and nonposition parts, we may assume that f is nonnegative. Then we know that $f = \lim f_k$ where each f_k, $k \in \mathbb{N}$, is a simple function. By (a), given $\varepsilon > 0$ there is a closed set C_k such that $\mathfrak{m}(A \setminus C_k) < \varepsilon/2^k$ and f_k is continuous on C_k. Let $K_1 = \bigcap_{k=1}^{\infty} C_k$. Show that $\mathfrak{m}(A \setminus K_1) < \varepsilon$. Use Egoroff's theorem to show that there exists a set $K_2 \subset K_1$ with $\mathfrak{m}(K_1 \setminus K_2) < \varepsilon$ and $f_k \to f$ uniformly on K_2. Conclude that f is continuous on K_2.
   (c) Now find a closed set $C \subset K_2$ such that $\mathfrak{m}(K_2 \setminus C) < \varepsilon$. Show that $\mathfrak{m}(A \setminus C) < 3\varepsilon$ and the restriction of f to C is a continuous function.

9. A sequence $\{f_n\}$ of a.e. real-valued, measurable functions is said to be **convergent in measure** if there is a measurable function f such that for each $\varepsilon > 0$,
$$\lim_{n\to\infty} \mu\Big(\{x\,;\,|f_n(x)-f(x)| \geq \varepsilon\}\Big) = 0.$$
Prove that if $\{f_n\}$ converges in measure to a function f, then f is a.e. real-valued, and if it converges to another function g in measure, then $f = g$ a.e.
10. Prove that if $f_n \to f$ a.u., then $f_n \to f$ in measure. From Egoroff's theorem prove that if X has finite measure, then any sequence $\{f_n\}$ of a.e. real-valued, measurable functions that converges a.e. to an a.e. real-valued, measurable function f also converges to f in measure.
11. A sequence $\{f_n\}$ of a.e. real-valued, measurable functions is said to be **Cauchy in measure** if for any $\varepsilon > 0$,
$$\mu\Big(\{x\,;\,|f_n(x)-f_m(x)| \geq \varepsilon\}\Big) \to 0, \qquad \text{as } n,m\to\infty.$$
Prove that if $f_n \to f$ in measure, then $\{f_n\}$ is Cauchy in measure.
12. Prove that if a sequence $\{f_n\}$ of a.e. real-valued, measurable functions is Cauchy in measure, then there is a subsequence $\{f_{n_k}\}$ and an a.e. real-valued, measurable function f such that $f_{n_k} \to f$ a.u. You may proceed as follows.
    (a) Show that there is an increasing sequence $n_1 < n_2 < \cdots$ such that
    $$\mu\Big(\{x\,;\,|f_n(x)-f_m(x)| \geq \varepsilon\}\Big) < \frac{1}{2^k}, \qquad \text{for all } n,m \geq n_k.$$
    (b) Let
    $$A_n = \bigcup_{k=n}^{\infty} \Big\{x\,;\,|f_{n_k}(x)-f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\Big\}.$$
    Show that $\{f_{n_k}\}$ is a Cauchy sequence of bounded functions on the set A_n^c. Deduce that there is a real-valued measurable function f on $A = \bigcup_{n=1}^{\infty} A_n^c$ such that $\{f_{n_k}\}$ converges uniformly to f on each A_n^c.
    (c) Define f to be zero on A^c. Show that $f_n \to f$ a.u.
13. Prove the following theorem.

Theorem 1.4 (Completeness for convergence in measure). *If $\{f_n\}$ is a sequence of a.e. real-valued, measurable functions that is Cauchy in measure, then there exists an a.e. real-valued, measurable function f such that $f_n \to f$ in measure.*

Suggestion: Use the previous problem to define f, and then prove and use the 'set-theoretic triangle inequality':
$$\{x\,;\,|f_n(x)-f(x)| \geq \varepsilon\} \subset \Big\{x\,;\,|f_n(x)-f_{n_k}(x)| \geq \frac{\varepsilon}{2}\Big\} \cup \Big\{x\,;\,|f_{n_k}(x)-f(x)| \geq \frac{\varepsilon}{2}\Big\}.$$

2. INTEGRATION OF MEASURABLE FUNCTIONS

Exercises 2.1.

1. Let f be a nonnegative real-valued measurable function on X. Prove that
$$\int f\,d\mu = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{k}{2^n}\,\mu(A_{nk}), \tag{2.1}$$

where $A_{nk} = f^{-1}(k/2^n, (k+1)/2^n]$. Now suppose that f is a real-valued integrable function. Prove that for any measurable set A with $\mu(A) < \infty$, we have

$$\int_A f \, d\mu = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \, \mu(A_{nk} \cap A), \tag{2.2}$$

where $A_{nk} = f^{-1}(k/2^n, (k+1)/2^n]$. The sum on the right is defined by

$$\sum_{k=-\infty}^{\infty} \frac{k}{2^n} \, \mu(A_{nk} \cap A) = \lim_{\ell\to\infty} \sum_{k=-\ell}^{\ell} \frac{k}{2^n} \, \mu(A_{nk} \cap A).$$

The condition $\mu(A) < \infty$ is needed: Can you give a very simple example of an integrable function on $\mathbb{R}$ such that the formula (2.2) is not true?

2. Using the explicit formula (2.1), compute $\int_0^1 x^2 \, dx$ and $\int_0^1 1/\sqrt{x} \, dx$.
3. (a) Let $\# : \mathscr{P}(\mathbb{N}) \longrightarrow [0,\infty]$ be the counting measure. Prove that any extended real-valued function f on $\mathbb{N}$ is measurable, and is integrable if and only if the series $\sum_{n=1}^{\infty} |f(n)|$ is convergent, in which case

   $$\int f \, d\# = \sum_{n=1}^{\infty} f(n).$$

   (b) Let $\mu : \mathscr{P}(\mathbb{N}\times\mathbb{N}) \longrightarrow [0,\infty]$ be the counting measure on $\mathscr{P}(\mathbb{N}\times\mathbb{N})$. Prove that any extended real-valued function f on $\mathbb{N}\times\mathbb{N}$ is measurable. Let $n_1, n_2, \ldots$ be any ordering of the countable set $\mathbb{N}\times\mathbb{N}$. Prove that given any nonnegative extended real-valued function f on $\mathbb{N}\times\mathbb{N}$, we have

   $$\int f \, d\mu = \sum_{k=1}^{\infty} f(n_k).$$

   This shows that for any sequence of nonnegative extended real numbers $\{a_{mn}\}$, the sum $\sum_{(m,n)} a_{mn}$ is defined independent of the way the sum is added. Use this to show that iterated sums are equal: $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}$. Note we have already used this fact many times.
4. Let g be a nonnegative measurable function and define $\mu_g : \mathscr{S} \longrightarrow [0,\infty]$ by

   $$\mu_g(A) = \int \chi_A \, g \, d\mu, \qquad \text{for all } A \in \mathscr{S}.$$

   (a) Prove that $\mu_g : \mathscr{S} \longrightarrow [0,\infty]$ is a measure.
   (b) Prove that for any nonnegative measurable function f, we have

   $$\int f \, d\mu_g = \int f \, g \, d\mu. \tag{2.3}$$

   (c) Given a measurable function f, prove that $f \in L^1(X, \mu_g)$ if and only if $fg \in L^1(X,\mu)$, in which case (2.3) holds.
   (d) Let g be a continuously differentiable nondecreasing function on $\mathbb{R}$ and denote by "g" the set function $g : \mathscr{I}^1 \longrightarrow [0,\infty)$ defined by $g(a,b] = g(b) - g(a)$. Given a Lebesgue measurable function f, prove that $f \in L^1(\mathbb{R}, g)$ if and only if $fg' \in L^1(\mathbb{R}, \mathfrak{m})$, in which case

   $$\int f \, dg = \int f \, g' \, dx.$$

5. Prove that Fatou's Lemma implies the Monotone Convergence Theorem.

6. In this problem we shall prove that Lebesgue integration over $\mathbb{R}^n$ is translation invariant in the sense that given any Lebesgue integrable function f on $\mathbb{R}^n$ and point $a \in \mathbb{R}^n$, we have

$$\int f(x+a)\,dx = \int f(x)\,dx. \tag{2.4}$$

Prove that if f is Lebesgue measurable, then $f(x+a)$ is a Lebesgue measurable function of x. Prove (2.4) for characteristic functions of Lebesgue measurable sets, then for nonnegative simple functions, then for nonnegative Lebesgue measurable functions via the Monotone Convergence Theorem, then finally for any Lebesgue integrable function. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a nonsingular linear transformation. Prove that if f is Lebesgue measurable, then $f \circ T$ is also Lebesgue measurable, and if f is Lebesgue integrable, then

$$\int f(x)\,dx = |\det T| \int f(T(x))\,dx.$$

In particular, Lebesgue integration is invariant under orthogonal transformations. We shall generalize these considerations later.

7. Let μ_1 and μ_2 be measures on $(X, \mathscr{S})$ and let $\mu = \mu_1 + \mu_2$.
   (a) Show that μ is a measure on $\mathscr{S}$.
   (b) Prove that a measurable function f is μ_i-integrable for $i = 1, 2$ if and only if f is μ-integrable, in which case

$$\int f\,d\mu = \int f\,d\mu_1 + \int f\,d\mu_2.$$

8. In this problem, we prove that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. Show that it suffices to prove that $\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$.
   (a) Taking the binomial expansion of $(1-x^2)^{-1/2}$ near $x = 0$, prove that

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{x^{2n+1}}{2n+1},$$

   and show that this series is valid for all x in $(-1, 1)$.[1]
   (b) Substituting $x = \sin t$ into this expression, obtain

$$t = \sin t + \sum_{n=1}^{\infty} \frac{1}{2n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \sin^{2n+1} t, \tag{2.5}$$

   valid for $-\pi/2 < t < \pi/2$.
   (c) Integrating the series (2.5) on the interval $0 < t < \pi/2$, prove that

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \int_0^{\pi/2} \sin^{2n+1} t\,dt.$$

   Make sure to justify this term-by-term integration. Using this expansion derive the sum $\pi^2/8 = \sum_{n=0}^{\infty} 1/(2n+1)^n$.

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[1] Using 'Raabe's test', one can in fact show that this series converges uniformly for x in $[-1, 1]$, but we won't need this fact.

3. PROPERTIES OF THE INTEGRAL

Exercises 3.1.

1. Let f be integrable. Prove the following properties of the integral.
   (a) If $f \geq 0$ a.e. and $A \subset B$ are measurable sets, then $\int_A f\,d\mu \leq \int_B f\,d\mu$.
   (b) If $f \geq a > 0$ a.e. on a measurable set A, then $\mu(A) < \infty$.
   (c) If $a \leq f \leq b$ on a measurable set A, then
   $$a\,\mu(A) \leq \int_A f\,d\mu \leq b\,\mu(A).$$
   (d) If $A \subset B$ are measurable and $\mu(B \setminus A) = 0$, then $\int_A f\,d\mu = \int_B f\,d\mu$.
2. A measurable function g is said to be **essentially bounded** if there is a real number M such that $|g(x)| \leq M$ for a.e. $x \in X$. If f is integrable and g is essentially bounded, prove that $f\,g$ is integrable. If X has finite measure, prove that any essentially bounded function is integrable.
3. Use the Dominated Convergence Theorem to establish the following result: If $\{f_n\}$ is a sequence of integrable functions such that
   $$\sum_{n=1}^{\infty} \int |f_n|\,d\mu < \infty,$$
   then $f(x) = \sum_{n=1}^{\infty} f_n(x)$ $(= \lim_{m\to\infty} \sum_{n=1}^{m} f_n(x))$ converges a.e., and
   $$\int f\,d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu.$$
4. The previous problem is used throughout applied mathematics. For example, the **Laplace transform** of a measurable function f on $[0,\infty)$ is defined by
   $$\mathscr{L}(f)(s) = \int_0^{\infty} e^{-sx} f(x)\,dx,$$
   provided that $e^{-sx}f(x)$ is integrable in x for all $s > c$ for some constant $c \geq 0$. For instance, according to Problem 7 below, we have $\mathscr{L}(x^n)(s) = n!/s^{n+1}$.
   (a) Let $f(x) = \sum_{n=0}^{\infty} a_n\,x^n$ be a power series such that $\sum_{n=0}^{\infty} |a_n|\,n!/s^{n+1}$ converges for $s > c$ for some $c \geq 0$. Prove that the Laplace transform of f exists for all $s > c$ and $\mathscr{L}(f)(s) = \sum_{n=0}^{\infty} a_n\,n!/s^{n+1}$ for all $s > c$.
   (b) Conversely, suppose that $\sum_{n=0}^{\infty} |a_n|\,n!/s^{n+1}$ converges for $s > c$ for some $c \geq 0$. Prove that $f(x) = \sum_{n=0}^{\infty} a_n\,x^n$ converges for a.e. $x \in [0,\infty)$ and $\mathscr{L}(f)(s) = \sum_{n=0}^{\infty} a_n\,n!/s^{n+1}$ for all $s > c$.
   (c) Using (a), find the Laplace transform of $\sin x$. (Answer: $1/(s^2+1)$).
   (d) Let $F(s) = 1/(s^2-1)$ where $s > 1$. Using (b), find a function f with $\mathscr{L}(f)(s) = F(s)$ for $s > 1$. (Answer: $\sinh(x)$.)
5. Let f be an integrable function. Prove that the set function $A \longmapsto \int_A f\,d\mu$ defined on measurable sets is continuous in the sense that given any $\varepsilon > 0$, there is a $\delta > 0$ such that
   $$\int_A |f|\,d\mu < \varepsilon$$
   for all measurable set A with $\mu(A) < \delta$. Suggestion: First prove this for bounded functions, then approximate $|f|$ by a nondecreasing sequence of simple functions, which are certainly bounded.
6. Prove the following General Dominated Convergence Theorem.

Theorem 3.2 (General Dominated Convergence Theorem). *Let f be a measurable function and let $\{f_n\}$ be a sequence of measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ for a.e. x and $|f_n| \le g_n$ a.e. for some integrable functions g_n. Suppose that $\lim_{n\to\infty} g_n(x)$ exists a.e. and*

$$\int \lim g_n \, d\mu = \lim \int g_n \, d\mu.$$

Then f and each f_n are integrable, and

$$\int f \, d\mu = \lim \int f_n \, d\mu.$$

7. Using the techniques outlined in lecture, we evaluate some integrals that occur in applications. You may use the Fundamental Theorem of Calculus.
   (a) Show that $\int_0^\infty e^{-tx}\, dx = 1/t$. Using this fact, prove that
   $$\int_0^\infty x^n \, e^{-tx} \, dx = \frac{n!}{t^{n+1}}, \quad n = 1, 2, \ldots.$$
   (b) Show that $\int_0^\infty e^{-t\,x^2}\, dx = \frac{1}{2}\sqrt{\pi/t}$ for $t > 0$. Using this fact, prove that
   $$\int_0^\infty x^{2n} \, e^{-x^2} \, dx = \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1}} \sqrt{\pi}, \quad n = 1, 2, \ldots.$$
   (c) Evaluate $\int_0^\infty x \, e^{-t\,x^2}\, dx$. Using this fact, prove that
   $$\int_0^\infty x^{2n+1} \, e^{-x^2} \, dx = \frac{n!}{2}, \quad n = 1, 2, \ldots.$$
   (d) Let
   $$F(t) = \int_0^\infty e^{-x^2 - t^2/x^2} \, dx, \quad t > 0.$$
   Compute $F'(t)$ and use this to prove that
   $$\int_0^\infty e^{-x^2 - t^2/x^2} \, dx = \frac{\sqrt{\pi}}{2} e^{-2t}, \quad t > 0.$$

8. (Jensen's inequality) We now prove Jensen's inequality for convex functions.
   (a) Let $I = (a, b)$ where $-\infty \le a < b \le \infty$. A function $\varphi : I \longrightarrow \mathbb{R}$ is said to be **convex** if
   $$\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y)$$
   for all $x, y \in I$ and $0 \le t \le 1$. Show that if φ is graphed and z is any point between x and y, then the point $(z, \varphi(z))$ lies on or below the line joining the points $(x, \varphi(x))$ and $(y, \varphi(y))$.
   (b) Show that φ is convex if and only if for all $a < x < y < z < b$, we have
   $$\frac{\varphi(y) - \varphi(x)}{y - x} \le \frac{\varphi(z) - \varphi(y)}{z - y}. \tag{3.1}$$
   (c) Suppose that φ is differentiable on I. Using (3.1), prove that φ is convex if and only if φ' is a nondecreasing function. In particular, any exponential function a^x where $a > 1$ is a convex function on $\mathbb{R}$.
   (d) Suppose that φ is convex on I. Using (3.1) prove that φ is Lipschitz on any closed interval of I, that is, given any closed interval $J \subset I$ there is a constant M such that $|\varphi(x) - \varphi(y)| \le M|x - y|$ for all $x, y \in J$. In particular, any convex function is continuous.

(e) Let φ be a convex function on I.

Theorem 3.3. *Let A be a measurable set with measure 1 and let f be an measurable function on A such that $f(x) \in I$ for all $x \in A$. Then*

$$\varphi\Big(\int_A f\,d\mu\Big) \le \int_A \varphi\circ f\,d\mu \quad \textit{Jensen's Inequality.}$$

You may proceed as follows: Let $y = \int_A f\,d\mu$. Show that $y \in I$. Let α be the supremum over $x \in (a, y)$ of the left-hand side of (3.1). Show that

$$\varphi(z) \ge \varphi(y) + \alpha(z - y)$$

for all $z \in I$. In particular, this inequality holds for $z = f(x)$ with $x \in A$. Now set $z = f(x)$ and integrate both sides of the inequality over A. Note that $\varphi\circ f$ is measurable (why?).

(f) Let $X = \{1, 2, \dots, n\}$ and given $\alpha_1, \dots, \alpha_n \ge 0$, define $\mu_\alpha : \mathscr{P}(X) \longrightarrow [0,\infty]$ by $\mu(\{j_1, \dots, j_k\}) = \alpha_{j_1} + \cdots + \alpha_{j_k}$. Prove that μ_α is a measure. Suppose now that $\mu(X) = 1$, that is, $\sum \alpha_k = 1$. Using Jensen's Inequality with $\varphi(x) = a^x$ where $a > 1$, prove that for any positive numbers $a_1, \dots, a_n$,

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \le \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n.$$

Suggestion: Let $f(k) = \log_a(a_k)$. In particular, if $\alpha_k = 1/n$ for each k, deduce that the geometric mean of n nonnegative real numbers never exceeds their arithmetic mean, that is, for any nonnegative numbers $a_1, \dots, a_n$,

$$(a_1\, a_2 \cdots a_n)^{1/n} \le \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

4. Lebesgue vs. Riemann integration

Exercises 4.1.

1. Let f be a unbounded real-valued function on an interval $[a, b]$. Prove that f cannot be Riemann integrable.
2. If f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(x)\,dx$, prove that $F'(x) = f(x)$ for a.e. $x \in [a, b]$.
3. We prove that a function is continuous almost everywhere if and only if it is has a left-hand limit almost everywhere.
   (a) Let f be a bounded real-valued function on an interval I. Given a point $a \in I$, we define the **oscillation of f at a** by

   $$\operatorname{osc}(f, a) = \lim_{\delta\to 0}\Big[\sup\{f(x)\,;\,|x - a| < \delta\} - \inf\{f(x)\,;\,|x - a| < \delta\}\Big].$$

   Let D denote the set of discontinuities of f. Prove that $D = \bigcup_{n=1}^{\infty} D_n$ where $D_n = \{x\,;\,\operatorname{osc}(f, x) > 1/n\}$.
   (b) Let L denote the set of points where f has a left-hand limit. Prove that every point of $D_n \cap L$ is the right endpoint of an open interval that contains no point of $D_n \cap L$. Conclude that $D_n \cap L$ must be countable, and hence $D \cap L$ must be countable.
   (c) Now prove that a bounded function on an interval is continuous a.e. if and only if it has a left-hand limit a.e.

4. The next problem will require the following fact: Prove that any monotone function on a closed interval has at most a countable number of discontinuities. Suggestion: Since $\mathbb{R}$ is a countable disjoint union of intervals, we may assume that the interval is bounded. Let g be a nondecreasing function on the interval. Prove that g is bounded. Note that g is discontinuous at a point x if and only if $d(x) = \lim_{y \to x+} g(y) - \lim_{y \to x-} g(y) > 0$. Let $D_n = \{x\,;\, d(x) > 1/n\}$. Prove that D_n is a finite set and that the set of discontinuities of g is $\bigcup_{n=1}^{\infty} D_n$.
5. Let g be a nondecreasing right-continuous function on an interval $[a, b]$. Then we know that the function g induces a measure $g : \mathscr{M}_g \longrightarrow [0, \infty]$, where $\mathscr{M}_g$ denotes the g-measurable sets. In this problem we prove the following theorem.

   Theorem 4.2. *A bounded function on a finite interval is Riemann-Stieltjes integrable with respect to g if and only if it is continuous g-a.e., that is, everywhere except perhaps on a set with g-measure zero, in which case the function is also (Lebesgue) g-integrable and the two notions of integral agree.*

   Let f be a bounded function on an interval $[a, b]$. You may proceed as follows:
   (a) Prove that f is Riemann-Stieltjes integrable if and only if for any sequence of nondecreasing partitions $\mathscr{P}_n$,
   $$\int_a^b L_P\, dg = \mathscr{R}\int_a^b f\, dg = \int_a^b U_P\, dg,$$
   where L_P and U_P denote the limits of the monotone sequences of lower and upper functions, respectively, of f with respect to g, and where P denote the union $P = \bigcup_{n=1}^{\infty} \mathscr{P}_n$. The integrals on the far left and right denote (Lebesgue) integrals with respect to the measure g.
   (b) The above equality holds if and only if $L_P = f = U_P$ g-a.e. Let D denote the set of discontinuities of g. Prove that $L_P = f = U_P$ g-a.e. if and only if $L_P(x) = f(x) = U_P(x)$ at each $x \in D$ and for g-a.e. on $[a, b] \setminus D$.
   (c) Finally, prove that $L_P = f = U_P$ g-a.e. if and only if $L_P(x) = f(x) = U_P(x)$ at each $x \in D$ and for g-a.e. on $[a, b] \setminus D$, if and only if f is continuous g-a.e. on $[a, b] \setminus D$ and continuous at each $x \in D$. Suggestion: To prove that f is continuous at $x \in D$, choose the partitions such that $x \notin P$.

5. Approximation theory in L^1

Exercises 5.1.

1. (Cf. Problem 6 in Exercise 2.1.) Given a Lebesgue integrable function f on $\mathbb{R}^n$ and a point $a \in \mathbb{R}^n$, define $f_a(x) = f(x + a)$. We shall prove that $f_a \to f$ in L^1 as $a \to 0$, that is, $\|f_a - f\|_{L^1} \to 0$ as $a \to 0$.
   (a) If f is the characteristic function of a left-half open box in $\mathscr{I}^n$, prove that $\|f_a - f\|_{L^1} \to 0$ as $a \to 0$.
   (b) If f is an $\mathscr{E}^n$-simple function, prove that $\|f_a - f\|_{L^1} \to 0$ as $a \to 0$.
   (c) Prove that $\|f_a - f\| \to 0$ as $a \to 0$ for any Lebesgue integrable function f.
2. We henceforth work in a general measure space. A sequence $\{f_n\}$ of integrable functions is said to be **Cauchy in L^1 (or Cauchy in the mean)** if
   $$\|f_n - f_m\|_{L^1} \to 0 \qquad \text{as } n, m \to \infty.$$
   Let $\{f_n\}$ be a sequence of integrable functions that is Cauchy in L^1 and suppose that there is a subsequence $\{f_{n_k}\}$ and an integrable function f such that $f_{n_k} \to f$ in L^1. Prove that the whole sequence f_n converges to f in L^1.

3. Prove that if a sequence $\{f_n\}$ of integrable functions is Cauchy in L^1, then there exists a real number M such that $\|f_n\|_{L^1} \leq M$ for each $n = 1, 2, 3, \ldots$.
4. Prove the **Tchebychev Inequality**: Let $f \in L^1(X)$ and given $\varepsilon > 0$, let $A_\varepsilon = \{x\,;\, |f(x)| \geq \varepsilon\}$. Then
$$\mu(A_\varepsilon) \leq \frac{\|f\|_{L^1}}{\varepsilon}.$$
Tchebychev's Inequality has two applications that we explore in the next two problems. We refer to Problems 9 and 11 in Exercises 1.1 for the definition of a sequence of measurable functions converging and being Cauchy in measure.
5. Let $f_n \to f$ in L^1. Prove that $f_n \to f$ in measure.
6. A sequence $\{f_n\}$ of integrable functions is said to be **Cauchy in** L^1 if
$$\|f_n - f_m\|_{L^1} \to 0 \qquad \text{as } n, m \to \infty.$$
Prove that if a sequence $\{f_n\}$ of integrable functions is Cauchy in L^1, then it is Cauchy in measure. In particular, according to Problem 13 in Exercises 1.1, given that $\{f_n\}$ is Cauchy in L^1, there exists a subsequence $\{f_{n_k}\}$ and an a.e. real-valued, measurable function f such that $f_{n_k} \to f$ a.u., and hence a.e. (See Problem 6 in Exercises 1.1).
7. Prove the following theorem.

Theorem 5.2 (Completeness for convergence in the mean). *If $\{f_n\}$ is a sequence of integrable functions that is Cauchy in L^1, then there exists an integrable function f such that $f_n \to f$ in L^1.*

You may proceed as follows.

(a) By Problem 6, we know that there is a subsequence $\{f_{n_k}\}$ and a measurable function f such that $f_{n_k} \to f$ a.e. Prove that f is in fact integrable, that is, $\int |f|\, d\mu < \infty$. Suggestion: Use Problem 3 and Fatou's Lemma.
(b) Let $\varepsilon > 0$ and choose N such that $\|f_{n_k} - f_{n_\ell}\|_{L^1} \leq \varepsilon$ for all $k, \ell \geq N$. Prove that $\|f - f_{n_\ell}\|_{L^1} \leq \varepsilon$ for each $\ell \geq N$. Suggestion: Again use Fatou's Lemma.
(c) Now invoke Problem 2 to prove that $f_n \to f$ in L^1.

6. Integration of complex-valued functions

Exercises 6.1.

1. Prove that a complex-valued function $f : X \longrightarrow \mathbb{C}$ is measurable if and only if $f^{-1}(A)$ is measurable for every open set $A \subset \mathbb{C}$.
2. (Averaging Theorem) Let X have finite measure. Let f be integrable and suppose that there is a closed set $C \subset \mathbb{C}$ such that for every measurable set $A \subset X$ with $\mu(A) > 0$, the averages of f over A are in C, that is,
$$\frac{1}{\mu(A)} \int_A f\, d\mu \in C. \tag{6.1}$$
Prove that $f(x) \in C$ for a.e. $x \in X$. Suggestion: Since C is closed, C^c is open, and hence is a countable union of open balls. Thus, it suffices to prove that $\mu(f^{-1}(B)) = 0$ where B is an open ball in C^c. Given such a ball, suppose that $A = f^{-1}(B)$ has positive measure. Using (6.1), derive a contradiction.