ON A GENERALIZATION OF THE DEFINITE INTEGRAL BY MR. H. LEBESGUE.

SUR UNE GÉNÉRALISATION DE L'INTÉGRALE DÉFINIE. COMPTES RENDUS DE L'ACADEMIE DES SCIENCES (1901), PP. 1025–1028.

(ROUGH TRANSLATION — ALL ERRORS ARE DUE TO PAUL LOYA!)

In the case of continuous functions, the notions of the integral and antiderivatives are identical. Riemann defined the integral of certain discontinuous functions, but all derivatives are not integrable in the sense of Riemann. Research into the problem of antiderivatives is thus not solved by integration, and one can desire a definition of the integral including as a particular case that of Riemann and allowing one to solve the problem of antiderivatives.(1) To define the integral of an increasing continuous function

$$y(x) \quad (a \leq x \leq b)$$

one divides the interval (a, b) into subintervals and forms the sum of the quantities obtained by multiplying the length of each subinterval by one of the values of y when x is in the subinterval. If x is in the interval (a_i, a_{i+1}), y varies between certain limits m_i, m_{i+1}, and conversely if y is between m_i and m_{i+1}, x is between a_i and a_{i+1}. Of course, instead of giving the division of the variation of x, that is to say, to give the numbers a_i, one could have given the division of the variation of y, that is to say, the numbers m_i. From here there are two manners of generalizing the concept of the integral. One sees that the first (to be given the numbers a_i) leads to the definition given by Riemann and the definitions of the integral by upper and lower sums given by Mr. Darboux. Let us see the second. Let the function y range between m and M. Given

$$m = m_0 < m_1 < m_2 < \cdots < m_{p-1} < M = m_p$$

$y = m$ when x belongs to the set E_0; $m_{i-1} < y \leq m_i$ when x belongs to the set E_i.[1] We will define the measures λ_0, λ_i of these sets. Let us consider either of the two sums

$$m_0\lambda_0 + \sum m_i\lambda_i \quad ; \quad m_0\lambda_0 + \sum m_{i-1}\lambda_i;$$

[1] Translator's footnote: That is, Lebesgue defines $E_0 = y^{-1}(m) = \{x \in [a, b]\, ;\, y(x) = m\}$ and $E_i = y^{-1}(m_{i-1}, m_i] = \{x \in [a, b]\, ;\, m_{i-1} < y(x) \leq m_i\}$.

if, when the maximum difference between two consecutive m_i *tends to zero, these sums tend to the same limit independent of the* m_i *chosen, this limit will be, by definition, the integral of* y*, which will be known as integrable.*

Let us consider a set of points of (a, b); one can enclose in an infinite number of ways these points in an infinite number of intervals; the infimum of the sum of the lengths of the intervals is the measure of the set.[2] A set E is said to be *measurable* if[3] its measure together with that of the set of points not forming E gives the measure of (a, b).[(2)] Here are two properties of these sets: Given an infinite number of measurable sets E_i, the set of points which belong to at least one of them is measurable; if the E_i are such that no two have a common point, the measure of the set thus obtained is the sum of measures of the E_i. The set of points in common with all the E_i is measurable.[4]

It is natural to consider first of all functions whose sets which appear in the definition of the integral are measurable. One finds that: *if a function bounded in absolute value is such that for any* A *and* B*, the values of* x *for which* $A < y \leq B$ *is measurable, it is integrable* by the process indicated. Such a function will be called *summable.* The integral of a summable function lies between the lower integral and the upper integral.[5] It follows that *if an integrable function is summable in the sense of Riemann, the integral is the same with the two definitions.* However, *any integrable function in the sense of Riemann is summable*, because the set of all its points of discontinuity has measure zero, and one can show that if, by omitting the set of values of x of measure zero, what remains is a set at each point of which the function is continuous, this function is summable. This property makes it possible to immediately form nonintegrable functions in the sense of Riemann and, however, are summable functions. Let $f(x)$ and $\varphi(x)$ be two continuous functions, $\varphi(x)$ not always zero; a function which

[2] Translator's footnote: Denoting by $\mathfrak{m}^*(E)$ the measure of a set $E \subseteq (a, b)$, Lebesgue is defining $\mathfrak{m}^*(E) := \inf\{\sum_i \ell(I_i)\,;\, E \subseteq \bigcup_i I_i\}$ where $I_i = (a_i, b_i]$ and $\ell(I_i) = b_i - a_i$. It's true that Lebesgue doesn't specify the types of intervals, but it doesn't matter what types of intervals you choose to cover E with (I chose left-half open ones because of my upbringing).

[3] Translator's footnote: Lebesgue is defining E to be measurable if $\mathfrak{m}^*(a, b) = \mathfrak{m}^*(E) + \mathfrak{m}^*((a, b) \cap E^c)$.

[4] Translator's footnote: Lebesgue is saying that if the E_i are measurable, then $\bigcup_i E_i$ is measurable, if the E_i are pairwise disjoint, then $\mathfrak{m}^*(\bigcup_i E_i) = \sum_i \mathfrak{m}^*(E_i)$, and finally, that $\bigcap_i E_i$ is measurable. The complement of a measurable set is, almost by definition, measurable, so Lebesgue is basically saying that the collection of all measurable sets forms a σ-algebra.

[5] Translator's footnote: Lower and upper integrals in the sense of Darboux.

does not differ from $f(x)$ at the points of a set of measure zero that is everywhere dense and which at these points is equal to $f(x) + \varphi(x)$ is summable without being integrable in the sense of Riemann. *Example:* The function equal to 0 if x is irrational, equal to 1 if x is rational. This process of forming combinations shows that the summable functions are far superior than the continuous ones. Here are two properties of these types of functions:

(1) *If f and φ are summable, $f + \varphi$ is and the integral of $f + \varphi$ is the sum of the integrals of f and of φ.*

(2) *If a sequence of summable functions has a limit, it is a summable function.*

The collection of summable functions obviously contains $y = k$ and $y = x$; therefore, according to (1), it contains all the polynomials and, according to (2), it contains all its limits, therefore it contains all the continuous functions, that is to say, the functions of first class (see Baire, *Annali di Matematica*, 1899), it contains all those of second class, etc. In particular, any derivative, bounded in absolute value, being of first class, is summable, and one can show that its integral, considered as function of its higher limit, is an antiderivative. Here is a geometrical application: if $|f'|$, $|\varphi'|$, $|\psi'|$ are bounded, the curve has a length given by the integral of $\sqrt{(f'^2 + \varphi'^2 + \psi'^2)}$. If $\varphi = \psi = 0$, one obtains the total variation of the function f of bounded variation. If f', φ', ψ' do not exist, one can obtain an almost identical theorem by replacing the derivatives by the Dini derivatives.

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Footnotes:

(1) These two conditions imposed *a priori* on any generalization of the integral are obviously compatible, because any integrable derivative, in the sense of Riemann, has as an integral one of its antiderivatives.

(2) If one adds to this collection suitably selected sets of measure zero, one obtains the measurable sets in the sense of Mr. Borel (*Leçons sur la théorie des fonctions*).