ON THE PROBLEM OF THE MEASURE OF THE SET OF POINTS OF A STRAIGHT LINE BY G. VITALI.

SUL PROBLEMA DELLA MISURA DEI GRUPPI DI PUNTI DI UNA RETTA. BOLOGNA, TIP. GAMBERINI E PARMEGGIANI, 1905, PP. 231–235.

(ROUGH TRANSLATION — ALL ERRORS ARE DUE TO PAUL LOYA!)

The problem of the measure of the set of points of a straight r is to determine for every set A of points of r a real positive number $\mu(A)$, called the **measure** of A, so that:[1]

1) Two sets that coincide by a suitable rigid movement of one of them have the same measure.

2) The union of a finite or a countably infinite number of sets, which are pairwise disjoint, has as a measure the sum of the measures.

3) The measure of the set of all points of interval $(0, 1)$ is 1.

Let x be a point of r. The points of r that differ from x by a rational number, positive, negative, or zero, forms a countable set A_x. If A_{x_1} and A_{x_2} are two such sets, they have no common points or they coincide.

Consider the different sets A_x; considered as elements they form a set H. If P is any point of r, there is one and only one element of H in which P belongs.

Consider for each element α of H a point P_α in the interval $(0, 1/2)$ that belongs to α and denote by G_0 the set of points P_α. If ρ is any rational number, we will denote by G_ρ the set of $P_\alpha + \rho$.[2]

The sets G_ρ corresponding to different values of rational ρ are pairwise disjoint, there are also countably many of them, and by 1) they should have the same measure.

The sets

$$G_0, \quad G_{\frac{1}{2}}, \quad G_{\frac{1}{3}}, \quad G_{\frac{1}{4}}, \ldots$$

all lie in the interval $(0, 1)$, therefore their union should have a measure $m \leq 1$.

But it must be that

$$\begin{aligned} m &= \mu(G_0) + \sum_{n=2}^{\infty} \mu(G_{\frac{1}{n}}) \\ &= \lim_{n\to\infty} n \cdot \mu(G_0), \end{aligned}$$

and therefore

$$\mu(G_0) = 0.$$

[1]See Leçons sur l'integration etc. by H. Lebesgue p. 103. Paris, Gauthier-Villars, 1904.

[2]Translator's footnote: Vitali actually says $P_\alpha - \rho$, but I think this is a typo and it should be $P_\alpha + \rho$. We need the "+" in order that G_0, $G_{\frac{1}{2}}$, $G_{\frac{1}{3}}$, $G_{\frac{1}{4}}, \ldots$ all lie in $(0, 1)$.

But then the union of all G_ρ corresponding to different rational values ρ must also have measure zero. But this union is the set of all points of r and therefore should have infinite measure.

This suffices us to conclude: **The problem of the measure of the set of points of a straight line is impossible.**

Something could be objectionable about considering the set G_0. This can be fully justified if it is accepted that the continuum can be well ordered. For those who do not want to accept our result, it follows that: *the possibility of the problem of the measure of sets of points of a straight line and the well ordering of the continuum cannot coexist.*