2.3.2. The π-λ Theorem. Let $\mathscr{I}$ be a collection of sets (no conditions on $\mathscr{I}$ for the moment). A natural question to ask is: Is a probability measure on $\mathscr{S}(\mathscr{I})$ uniquely determined by its values on $\mathscr{I}$? More precisely, let

$$\mu : \mathscr{S}(\mathscr{I}) \to [0,1] \quad \text{and} \quad \nu : \mathscr{S}(\mathscr{I}) \to [0,1]$$

be two probability measures.

Question: If μ and ν are the same on $\mathscr{I}$, is $\mu = \nu$ on $\mathscr{S}(\mathscr{I})$?

Stated another way, if we denote by

$$\mathcal{L} = \{A \in \mathscr{S}(\mathscr{I})\,;\, \mu(A) = \nu(A)\}, \tag{2.15}$$

when we want to know if $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}$ (which implies $\mathscr{S}(\mathscr{I}) = \mathcal{L}$ since $\mathcal{L} \subseteq \mathscr{S}(\mathscr{I})$ by definition of $\mathcal{L}$). The easiest way to prove that $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}$ is to apply the familiar

> **Principle of Appropriate Sets:** If a set $\mathcal{L}$ is a σ-algebra and we can show that $\mathcal{L}$ contains the "appropriate sets," namely all sets in $\mathscr{I}$, then $\mathcal{L}$ must contain all of $\mathscr{S}(\mathscr{I})$.

So, let's try to prove that $\mathcal{L}$ is a σ-algebra. First, certainly, $\varnothing \in \mathcal{L}$. Second, if $A \in \mathcal{L}$, then $\mu(A) = \nu(A)$, so

$$\mu(A^c) = 1 - \mu(A) = 1 - \nu(A) = \nu(A^c),$$

implies that $A^c \in \mathcal{L}$. Is $\mathcal{L}$ closed under arbitrary countable unions? This isn't so obvious because measures only behave well on countable unions when the sets have certain properties, like being pairwise disjoint. For example, we can easily prove that $\mathcal{L}$ is closed under countable unions of pairwise disjoint sets, for if $A_1, A_2, \ldots$ are pairwise disjoint elements of $\mathcal{L}$, then $\mu(A_n) = \nu(A_n)$ for each n, so with $A = \bigcup_{n=1}^{\infty} A_n$, we have

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \nu(A).$$

To summarize, we've shown that $\mathcal{L}$ contains the empty set and is closed under complements and countable unions of pairwise disjoint sets. Thus, $\mathcal{L}$ is almost a σ-algebra but not quite.

Eugene Dynkin (1924–).

There is a name for collections of sets with these properties: A collection $\mathcal{L}$ of subsets of X is called a **λ-system** (λ for lattice) if $\mathcal{L}$

(1) contains the empty set;
(2) is closed under complements;
(3) is closed under countable unions of pairwise disjoint sets.

Now, $\mathcal{L}$, although an λ-system, is not necessarily a σ-algebra, so we cannot at this point conclude that $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}$. Fortunately, Theorem 2.17 below, called the π-λ theorem (π for product), tells us if we assume that $\mathscr{I}$ is closed under intersections (or "products"), then we can in fact conclude that $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}$. λ-systems and the π-λ theorem were first introduced in Eugene Borisovich Dynkin's (1924–) 1959 (Russian) book on Markov Processes (according to the supplementary notes in the 1961 English translation [**49**]).

Theorem 2.17 (**π-λ theorem**). *Let $\mathscr{I}$ be a collection of sets closed under intersections and let $\mathcal{L}$ be a collection of sets such that*

$$\mathscr{I} \subseteq \mathcal{L} \quad \text{and} \quad \mathcal{L} \text{ is a } \lambda\text{-system.}$$

Then $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}$.

Proof. Let $\mathcal{L}_0$ be the smallest[6] λ-system containing $\mathscr{I}$. Then $\mathcal{L}_0 \subseteq \mathcal{L}$. We shall prove that $\mathcal{L}_0$ is a σ-algebra. Then $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}_0$ since $\mathscr{S}(\mathscr{I})$ is the smallest σ-algebra containing $\mathscr{I}$, and hence $\mathscr{S}(\mathscr{I}) \subseteq \mathcal{L}$ as well. Our first step to show that $\mathcal{L}_0$ is a σ-algebra is by proving

Step 1: If $\mathcal{L}_0$ is closed under intersections, then it's a σ-algebra. To prove this, assume that $\mathcal{L}_0$ is closed under intersections and let $A_1, A_2, \ldots \in \mathcal{L}_0$; we have to show that $\bigcup_{n=1}^\infty A_n \in \mathcal{L}_0$. Observe that

$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n,$$

where $B_n = A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c$. By definition of λ-system, $\mathcal{L}_0$ is closed under complements and, by assumption, intersections, so $B_n \in \mathcal{L}_0$. Moreover, one can check that the B_n's are pairwise disjoint, so $\bigcup_{n=1}^\infty B_n \in \mathcal{L}_0$. This shows that $\mathcal{L}_0$ is a σ-algebra.

Step 2: Therefore, according to **Step 1**, to prove that $\mathcal{L}_0$ is a σ-algebra we just have to show that $\mathcal{L}_0$ is closed under intersections. For this reason, given any $A \in \mathcal{L}_0$, let us put

$$\mathcal{L}_A = \{B \in \mathcal{L}_0\,;\, A \cap B \in \mathcal{L}_0\};$$

our goal is to prove that $\mathcal{L}_0 \subseteq \mathcal{L}_A$. To prove this, we just have to show that $\mathcal{L}_A$ is a λ-system and it contains $\mathscr{I}$, this show that $\mathcal{L}_0 \subseteq \mathcal{L}_A$ because $\mathcal{L}_0$ is the smallest λ-system containing $\mathscr{I}$.

Given $A \in \mathcal{L}_0$ we show that $\mathcal{L}_A$ is a λ-system; we shall prove that $\mathcal{L}_A$ contains $\mathscr{I}$ in **Step 4** below. It's easy to check that $\varnothing \in \mathcal{L}_A$. Let $B \in \mathcal{L}_A$; we need to show that $B^c \in \mathcal{L}_A$, which means that $A \cap B^c \in \mathcal{L}_0$. To see this, observe that

$$A \cap B^c = A \cap (A \cap B)^c = \Big(A^c \cup (A \cap B)\Big)^c.$$

Since $\mathcal{L}_0$ is closed under complements, $A^c \in \mathcal{L}_0$ and since $A \cap B \in \mathcal{L}_0$ (because $B \in \mathcal{L}_A$) and $\mathcal{L}_0$ is closed under unions of disjoint sets, we have $A^c \cup (A \cap B) \in \mathcal{L}_0$. By closedness under complements again, it follows that $A \cap B^c \in \mathcal{L}_0$. Thus, $B^c \in \mathcal{L}_0$. Finally, let $B_1, B_2, \ldots$ be pairwise disjoint elements of $\mathcal{L}_A$; we need to show that $B := \bigcup_{n=1}^\infty B_n \in \mathcal{L}_A$. To do so, observe that

$$A \cap B = \bigcup_{n=1}^\infty A \cap B_n.$$

By definition of $\mathcal{L}_A$, we have $A \cap B_n \in \mathcal{L}_0$ for each n and since $\mathcal{L}_0$ is closed under countable unions of disjoint sets, it follows that $A \cap B \in \mathcal{L}_0$. Thus, $B \in \mathcal{L}_A$.

Step 3: Given $I \in \mathscr{I}$, we claim that $\mathcal{L}_0 \subseteq \mathcal{L}_I$. Indeed, since $I \in \mathcal{L}_0$ also, by **Step 2** we know that $\mathcal{L}_I$ is a λ-system. Moreover, since $\mathscr{I}$ is closed under intersections, by definition of $\mathcal{L}_I$ it follows that $\mathscr{I} \subseteq \mathcal{L}_I$. Since $\mathcal{L}_0$ is the smallest λ-system containing $\mathscr{I}$, we get $\mathcal{L}_0 \subseteq \mathcal{L}_I$.

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[6] Copying the proof of Theorem 1.5 you can prove that any collection of subsets of a given set always has a smallest λ-system containing it.

Step 4: Finally, given $A \in \mathcal{L}_0$, we show that $\mathscr{I} \subseteq \mathcal{L}_A$. Let $I \in \mathscr{I}$; we must show that $I \in \mathcal{L}_A$, which means $A \cap I \in \mathcal{L}_0$. However, observe that

$$\begin{aligned} A \in \mathcal{L}_0 \quad &\Longrightarrow \quad A \in \mathcal{L}_I \quad \text{(by } \textbf{Step 3}\text{)} \\ &\Longrightarrow \quad A \cap I \in \mathcal{L}_0 \quad \text{(definition of } \mathcal{L}_I\text{)}. \end{aligned}$$

This completes our proof. □

Back to the original question that started this business! If $\mathscr{I}$ is closed under intersections, then the answer our question is "yes," two probability measures on $\mathscr{S}(\mathscr{I})$ that agree on $\mathscr{I}$ must agree on all of $\mathscr{S}(\mathscr{I})$. We leave you to check that the answer is also "yes" if we just assume that the measures are *finite-valued*, meaning that they never take on the value ∞, and if $\mu(X) = \nu(X)$. (Finiteness is used to prove that $\mathcal{L}$ is closed under complements.) Thus, we have the following

Corollary 2.18. *If $\mathscr{I}$ is a collection of sets closed under intersections and μ and ν are two finite-valued measures on $\mathscr{S}(\mathscr{I})$ with $\mu(X) = \nu(X)$ that agree on $\mathscr{I}$, then μ and ν agree on all of $\mathscr{S}(\mathscr{I})$.*