

2.1. Countably additive set functions on semirings and rings

In this section we study measures on semirings and rings. In particular, we show that Lebesgue measure and the infinite product measure on a sequence space both live up to their names: they are indeed measures.

2.1.1. Measures. An additive set function $\mu : \mathcal{S} \rightarrow [0, \infty]$ on a semiring \mathcal{S} (or ring, or σ -algebra, ...) is said to be **countably additive** if for any countable collection of pairwise disjoint sets A_1, A_2, A_3, \dots in \mathcal{S} such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$, we have

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A countably additive set function is called a **measure**. Here are a couple remarks about this definition. First, note that if \mathcal{S} is a finite collection of sets, μ is automatically countably additive (can you prove this?). In particular, all probability set functions where the sample space is finite are measures. So, countable additivity is a new idea only when \mathcal{S} is not finite. Second, note that if \mathcal{S} is a σ -algebra, then the assumption $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$ is automatically satisfied, otherwise we have to make this assumption. So, although we work with semirings and rings in this section, the natural domain for a measure is really a σ -algebra.

Later we shall prove that the additive set functions of geometry (Lebesgue measure) and those we have looked at in our probability examples (eg. infinite product measures) are all countably additive. However, lest you think that all finitely additive set functions are countably additive, consider the following example. (There are other examples in the exercises.)

Example 2.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the nondecreasing function defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Recall that the corresponding Lebesgue-Stieltjes set function is defined by $\mu_f(a, b] = f(b) - f(a)$. We know (by Proposition 1.22) that $\mu_f : \mathcal{S}^1 \rightarrow [0, \infty)$ is finitely additive. However, μ_f is not countably additive, for, consider the decomposition

$$(0, 1] = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right], \quad \text{or} \quad A = \bigcup_{n=1}^{\infty} A_n,$$

where $A = (0, 1]$ and $A_n = \left(\frac{1}{n+1}, \frac{1}{n} \right]$; observe that A_1, A_2, \dots are pairwise disjoint. For any $n \in \mathbb{N}$, we have

$$\mu_f(A_n) = f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) = 1 - 1 = 0,$$

so $\sum_{n=1}^{\infty} \mu_f(A_n) = 0$. However, $\mu_f(A) = f(1) - f(0) = 1$, so, as $1 \neq 0$,

$$\mu_f(A) \neq \sum_{n=1}^{\infty} \mu_f(A_n).$$

Thus $\mu_f : \mathcal{S}^1 \rightarrow [0, \infty)$ is not countably additive.

Recall from Theorem 1.27 that any additive set function $\mu : \mathcal{S} \rightarrow [0, \infty]$ on a semiring \mathcal{S} is **finitely subadditive**, meaning that if $A \in \mathcal{S}$ and $A \subseteq \bigcup_{n=1}^N A_n$ where $A_1, \dots, A_N \in \mathcal{S}$, then

$$\mu(A) \leq \sum_{n=1}^N \mu(A_n).$$

In general, we cannot replace the finite N with infinity. For example, in the above example we have $A \subseteq \bigcup_{n=1}^{\infty} A_n$ (in fact, we have $=$ rather than \subseteq), where $A = (0, 1]$ and $A_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$, but as $1 \notin A$, we have

$$\mu_f(A) \not\leq \sum_{n=1}^{\infty} \mu_f(A_n).$$

When we can replace N with infinity we have what's called countable subadditivity: An additive set function $\mu : \mathcal{S} \rightarrow [0, \infty]$ on a semiring \mathcal{S} is said to be **countably subadditive** if $A \subseteq \bigcup_{n=1}^{\infty} A_n$ where $A, A_1, A_2, \dots \in \mathcal{S}$ implies

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

In probability theory, this inequality is called **Boole's inequality** after George Boole (1815–1864). It turns out that countable additivity and countable subadditivity are intimately related.

THEOREM 2.1. *If $\mu : \mathcal{S} \rightarrow [0, \infty]$ is additive on a semiring, then*

$$\mu \text{ is countably additive} \iff \mu \text{ is countably subadditive};$$

that is, μ is a measure if and only if μ is countably subadditive.

PROOF. Suppose that μ is countably subadditive and let $A \in \mathcal{S}$ with $A = \bigcup_{n=1}^{\infty} A_n$ where $A_1, A_2, \dots \in \mathcal{S}$ are pairwise disjoint. In particular, $A \subseteq \bigcup_{n=1}^{\infty} A_n$, so by countable subadditivity,

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Also, since $A = \bigcup_{n=1}^{\infty} A_n$ we have $\bigcup_{n=1}^{\infty} A_n \subseteq A$, so by countable superadditivity,

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(A).$$

Thus, $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$, so μ is countably additive. We now prove the converse.

Suppose that μ is a countably additive. To show that μ is countably subadditive, let $A \in \mathcal{S}$ with $A \subseteq \bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{S}$ for each n ; we need to show that $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$. To prove this, we first intersect both sides of $A \subseteq \bigcup_{n=1}^{\infty} A_n$ with A to obtain $A = \bigcup_{n=1}^{\infty} (A \cap A_n)$. By the Fundamental Lemma of Semirings, there exists pairwise disjoint sets $\{B_{nm}\} \subseteq \mathcal{S}$ such that for each n , $B_{nm} \subseteq (A \cap A_n)$ are finite in number, and

$$A = \bigcup_n (A \cap A_n) = \bigcup_{n,m} B_{nm}.$$

By superadditivity, we have

$$\sum_m \mu(B_{nm}) \leq \mu(A \cap A_n) \leq \mu(A_n),$$

where we used that $\mu(A \cap A_n) \leq \mu(A_n)$ by monotonicity. Therefore, by countable additivity,

$$A = \bigcup_{n,m} B_{nm} \implies \mu(A) = \sum_n \sum_m \mu(B_{nm}) \leq \sum_n \mu(A_n).$$

□

2.1.2. Lebesgue and Lebesgue-Stieltjes set functions. We now prove that Lebesgue measure deserves to be called a measure.

THEOREM 2.2. *For any n , Lebesgue measure $\mathfrak{m} : \mathcal{I}^n \rightarrow [0, \infty)$ is a measure, that is, \mathfrak{m} is countably additive.*

PROOF. Recall that we denote a box $(p_1, q_1] \times \cdots \times (p_n, q_n]$ in \mathbb{R}^n by the notation $(p, q]$ where $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ are elements of \mathbb{R}^n . Of course, this box could be the empty set if any $p_k \geq q_k$. Given $r, s \in \mathbb{R}$, we let $(p+r, q+s]$ be the box determined by the n -tuples (p_1+r, \dots, p_n+r) and (q_1+s, \dots, q_n+s) . There are analogous notations for other types of boxes, closed, open, right-half open, etc.

To prove that \mathfrak{m} is a measure on \mathcal{I}^n , we shall instead prove that \mathfrak{m} is countably subadditive. The idea is to use a compactness argument and an “ $\frac{\varepsilon}{2^k}$ trick” to reduce *countable* subadditivity to *finite* subadditivity (and we know that \mathfrak{m} is finitely subadditive since it is finitely additive). Let $(a, b] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k]$ where $(a, b] \in \mathcal{I}^n$ and each $(a_k, b_k] \in \mathcal{I}^n$; we need to prove that $\mathfrak{m}(a, b] \leq \sum_{k=1}^{\infty} \mathfrak{m}(a_k, b_k]$. To do so, let $\varepsilon > 0$ and for each $k \in \mathbb{N}$, take a positive number $\delta_k > 0$ (later on in the proof we’ll choose these δ_k ’s in a nice way). Observe that

$$[a + \varepsilon, b] \subseteq (a, b] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k + \delta_k).$$

We now use compactness: Since the closed box $[a + \varepsilon, b]$ is compact, it is covered by a finite union of the open sets on the far right, say the first N of them,

$$[a + \varepsilon, b] \subseteq \bigcup_{k=1}^N (a_k, b_k + \delta_k).$$

Since $(a + \varepsilon, b] \subseteq [a + \varepsilon, b]$ and $(a_k, b_k + \delta_k] \subseteq (a_k, b_k + \delta_k)$, we conclude that

$$(a + \varepsilon, b] \subseteq \bigcup_{k=1}^N (a_k, b_k + \delta_k].$$

Now, \mathfrak{m} is finitely additive and hence finitely subadditive, so

$$\mathfrak{m}(a + \varepsilon, b] \leq \sum_{k=1}^N \mathfrak{m}(a_k, b_k + \delta_k] \leq \sum_{k=1}^{\infty} \mathfrak{m}(a_k, b_k + \delta_k],$$

and hence,

$$\mathfrak{m}(a + \varepsilon, b] \leq \sum_{k=1}^{\infty} \mathfrak{m}(a_k, b_k + \delta_k].$$

We now use the “ $\frac{\varepsilon}{2^k}$ trick”: Using the fact that for any $p, q \in \mathbb{R}^n$, $\mathbf{m}(p, q+r) - \mathbf{m}(p, q]$ is a continuous function of $r \in [0, \infty)$ and it vanishes at $r = 0$, we can take $\delta_k > 0$ small enough so that

$$\mathbf{m}(a_k, b_k + \delta_k] - \mathbf{m}(a_k, b_k] \leq \frac{\varepsilon}{2^k}.$$

Hence, $\mathbf{m}(a_k, b_k + \delta_k] \leq \frac{\varepsilon}{2^k} + \mathbf{m}(a_k, b_k]$, so

$$\sum_{k=1}^{\infty} \mathbf{m}(a_k, b_k + \delta_k] \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} + \sum_{k=1}^{\infty} \mathbf{m}(a_k, b_k].$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, we see that

$$\mathbf{m}(a + \varepsilon, b] \leq \varepsilon + \sum_{k=1}^{\infty} \mathbf{m}(a_k, b_k].$$

Finally, letting $\varepsilon \rightarrow 0$ and using the fact that $\mathbf{m}(a + \varepsilon, b]$ is a continuous function of ε , we obtain $\mathbf{m}(a, b] \leq \sum_{k=1}^{\infty} \mathbf{m}(a_k, b_k]$ as desired. \square

We now consider Lebesgue-Stieltjes set functions. In Proposition 1.22 we proved that any such set function is finitely additive. However, we saw in Example 2.1 that μ_f may not be countably additive, which was for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Note that in this example, f is not right continuous at 0. We shall prove that a Lebesgue-Stieltjes set function of a nondecreasing function f is a measure if and only if f is right-continuous (at every point).

THEOREM 2.3. *The Lebesgue-Stieltjes set function μ_f of a nondecreasing function f is a measure on \mathcal{I}^1 if and only if f is right-continuous (at every point). In particular, the Lebesgue-Stieltjes set function of a nondecreasing continuous function is a measure on \mathcal{I}^1 .*

PROOF. We break this proof into two steps.

Step 1: We first prove that if $(a, b] = \bigcup_{n=1}^{\infty} I_n$ is a union of pairwise disjoint nonempty sets $I_n \in \mathcal{I}^1$, then we can rewrite this union as

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

where $b_1 = b$, $a_n = b_{n+1}$ for $n = 1, 2, \dots$, and $a_n \rightarrow a$ as $n \rightarrow \infty$. In fact, since the proof is only so slightly different from the proof of Lemma 1.21 (assuming you did prove this lemma), we shall leave the details to you.

Step 2: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be nondecreasing. We first prove sufficiency. So suppose that f is right-continuous and let $(a, b] = \bigcup_{n=1}^{\infty} I_n$ be a union of pairwise disjoint nonempty sets $I_n \in \mathcal{I}^1$; we shall prove that $\mu_f(a, b] = \sum_{n=1}^{\infty} \mu_f(I_n)$. According to **Step 1** we can rewrite this union as

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

where $b_1 = b$, $a_n = b_{n+1}$ for $n = 1, 2, \dots$, and $a_n \rightarrow a$ as $n \rightarrow \infty$. Now observe that the following sum telescopes:

$$\sum_{n=1}^N \mu_f(a_n, b_n] = \sum_{n=1}^N (f(b_n) - f(a_n)) = f(b) - f(a_N).$$

Taking $N \rightarrow \infty$ and using the fact that $f(a_{N+1}) \rightarrow f(a)$, since f is right-continuous, we obtain $\sum_{n=1}^{\infty} \mu_f(I_n) = \mu_f(a, b]$.

We now prove necessity, so suppose that μ_f is countably additive and fix $x \in \mathbb{R}$. To prove that f is right-continuous at x we just have to prove that $f(x) = \lim_{n \rightarrow \infty} f(x_n)$ for any strictly decreasing sequence $x_1 > x_2 > x_3 > \dots$ with $\lim_{n \rightarrow \infty} x_n = x$ (why?). Consider the union

$$(x, x_1] = \bigcup_{n=1}^{\infty} (x_{n+1}, x_n]$$

and note that the sets $(x_{n+1}, x_n]$ are pairwise disjoint for different n . Since μ_f is assumed to be a measure,

$$\begin{aligned} f(x_1) - f(x) &= \mu_f(x, x_1] = \sum_{n=1}^{\infty} \mu_f(x_{n+1}, x_n] \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} (f(x_n) - f(x_{n+1})). \end{aligned}$$

The last sum telescopes to $\sum_{n=1}^{N-1} (f(x_n) - f(x_{n+1})) = f(x_1) - f(x_N)$ and we find that

$$f(x_1) - f(x) = f(x_1) - \lim_{N \rightarrow \infty} f(x_N).$$

Cancelling off $f(x_1)$ shows that $f(x) = \lim_{N \rightarrow \infty} f(x_N)$. \square

2.1.3. Extensions of measures from semirings to rings. In Theorem 2.5 below we show that any measure on a semiring can be extended to the generated ring. The primary example is extending Lebesgue measure from \mathcal{I}^n , the left-half open boxes, to the set of all elementary figures \mathcal{E}^n in \mathbb{R}^n . Before proving Theorem 2.5, we need the following result on double summations.

LEMMA 2.4. *For each $(m, n) \in \mathbb{N} \times \mathbb{N}$, let $a_{mn} \in [0, \infty]$ and let $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be a bijective function; therefore $f(1), f(2), f(3), \dots$ is a list of all elements of $\mathbb{N} \times \mathbb{N}$. Then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} a_{f(n)}.$$

PROOF. We remark that the double sum $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}$ means for each $m \in \mathbb{N}$, to sum the inner summation $\sum_{n=1}^{\infty} a_{mn}$. This gives a nonnegative extended real number for each $m \in \mathbb{N}$. We then sum all these numbers from $m = 1$ to ∞ . (The double sum $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$ has a similar meaning.) An alternative way to look at double sums are as follows. If $s_{mn} := \sum_{i=1}^m \sum_{j=1}^n a_{ij}$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_{mn} \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} s_{mn},$$

where the iterated limits, eg. $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_{mn}$, means for each $m \in \mathbb{N}$, take the inner limit first $\lim_{n \rightarrow \infty} s_{mn}$, then to take the outer limit $m \rightarrow \infty$ next. Let $L_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}$, $L_2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$, $L_3 = \sum_{n=1}^{\infty} a_{f(n)}$, and put

$$L := \sup \{s_{mn} ; m, n \in \mathbb{N}\};$$

we shall prove that

$$L_1 = L_2 = L_3 = L.$$

We first show that $L_1, L_2, L_3 \leq L$, then we prove the opposite inequality. For all m, n , we have, by definition of L , $s_{mn} \leq L$, so using the fact that limits preserve inequalities, we have

$$\lim_{n \rightarrow \infty} s_{mn} \leq L.$$

Now taking $m \rightarrow \infty$, we get

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_{mn} \leq L \implies L_1 \leq L.$$

Similarly, $L_2 \leq L$. Given $n \in \mathbb{N}$, we can choose m so that $f(1), \dots, f(n) \in \mathbb{N} \times \mathbb{N}$ are amongst the a_{ij} 's for $1 \leq i, j \leq m$. Then,

$$\sum_{i=1}^n a_{f(i)} \leq \sum_{i=1}^m \sum_{j=1}^m a_{ij} \leq L,$$

by definition of L . Taking $n \rightarrow \infty$ we get $L_3 \leq L$. Thus, $L_1, L_2, L_3 \leq L$. On the other hand, by definition of L_1 and L_2 , for any m, n we have

$$s_{mn} \leq L_1 \quad \text{and} \quad s_{mn} \leq L_2.$$

Thus, taking the supremum over all m, n , we see that $L \leq L_1$ and $L \leq L_2$. Also, given $m, n \in \mathbb{N}$, using that $f(1), f(2), f(3), \dots$ is a list of all elements of $\mathbb{N} \times \mathbb{N}$, we can choose $N \in \mathbb{N}$ so that $f(1), f(2), \dots, f(N)$ contain all the pairs (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence,

$$s_{mn} \leq \sum_{i=1}^N a_{f(i)} \leq L_3.$$

Taking the supremum over all m, n , we get $L \leq L_3$. □

THEOREM 2.5 (The Extension Theorem for Semirings). *An additive set function $\mu : \mathcal{S} \rightarrow [0, \infty]$ is a measure on a semiring \mathcal{S} if and only if its ring extension $\mu : \mathcal{R}(\mathcal{S}) \rightarrow [0, \infty]$ is a measure on $\mathcal{R}(\mathcal{S})$.*

PROOF. Assume that μ is countably additive on \mathcal{S} and using the same notation, let μ denote the extended set function on $\mathcal{R}(\mathcal{S})$. Let $A \in \mathcal{R}(\mathcal{S})$ and assume that $A = \bigcup_{m=1}^{\infty} B_m$ where the $B_m \in \mathcal{R}(\mathcal{S})$ are pairwise disjoint; we need to show that $\mu(A) = \sum_{m=1}^{\infty} \mu(B_m)$. Since μ is additive on $\mathcal{R}(\mathcal{S})$ and each B_m is a pairwise disjoint union of sets in \mathcal{S} (by Theorem 1.6), we may assume that $B_m \in \mathcal{S}$ for each m . Since $A \in \mathcal{R}(\mathcal{S})$, we can write $A = \bigcup_n A_n$ where the union is finite and the A_n 's are pairwise disjoint sets in \mathcal{S} , so

$$\bigcup_n A_n = \bigcup_m B_m,$$

since both sides equal A . Intersecting both sides with A_n and then with B_m give, respectively,

$$A_n = \bigcup_m (A_n \cap B_m) \quad , \quad B_m = \bigcup_n (A_n \cap B_m).$$

These unions express A_n and B_m as unions of pairwise disjoint sets in \mathcal{I} and hence,

$$\begin{aligned} \mu(A) &= \sum_n \mu(A_n) && \text{(by definition of } \mu \text{ on } \mathcal{R}(\mathcal{I})) \\ &= \sum_n \sum_m \mu(A_n \cap B_m) && (\mu \text{ is countably additive on } \mathcal{I}) \\ &= \sum_m \sum_n \mu(A_n \cap B_m) && \text{(Lemma 2.4)} \\ &= \sum_m \mu(B_m) && (\mu \text{ is countably additive on } \mathcal{I}). \end{aligned}$$

□

As an easy corollary, we obtain

COROLLARY 2.6 (Lebesgue and Lebesgue-Stieltjes measures).

- (1) For each n , Lebesgue measure \mathfrak{m} on \mathcal{I}^n extends uniquely to a measure on the ring of elementary figures \mathcal{E}^n .
- (2) The Lebesgue-Stieltjes measure μ_f on \mathcal{I}^1 of any right-continuous nondecreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ extends uniquely to a measure on the ring of elementary figures \mathcal{E}^1 .

2.1.4. Continuity of measures. We now give characterizations of measures in terms of continuity. A sequence of sets $\{A_n\}$, $n = 1, 2, \dots$, is **nondecreasing** if $A_n \subseteq A_{n+1}$ for each n , in which case, the **limit set** is by definition

$$\lim A_n := \bigcup_{n=1}^{\infty} A_n.$$

The sequence is **nonincreasing** if $A_n \supseteq A_{n+1}$ for each n , and in this case,

$$\lim A_n := \bigcap_{n=1}^{\infty} A_n.$$

An additive set function $\mu : \mathcal{R} \rightarrow [0, \infty]$ on a ring \mathcal{R} is said to be

- (1) **continuous from below** if for any nondecreasing sequence of sets $\{A_n\}$ in \mathcal{R} with limit set $A = \lim A_n \in \mathcal{R}$, we have

$$\mu(A) = \lim \mu(A_n).$$

- (2) μ is **continuous from above** if for any nonincreasing sequence of sets $\{A_n\}$ in \mathcal{R} with limit set $A = \lim A_n \in \mathcal{R}$ and $\mu(A_1) \neq \infty$, we have

$$\mu(A) = \lim \mu(A_n).$$

Note that in both cases above, the equality $\mu(A) = \lim \mu(A_n)$ can be written in the more suggestive notation:

$$\mu(\lim A_n) = \lim \mu(A_n),$$

which should ring a bell from your study of continuous functions; hence the term “continuous.” We call a set function μ **continuous** if it is continuous from below and continuous from above.

THEOREM 2.7. *If $\mu : \mathcal{R} \rightarrow [0, \infty]$ is an additive set function on a ring \mathcal{R} , then*

- (1) μ is a measure if and only if μ is continuous from below.

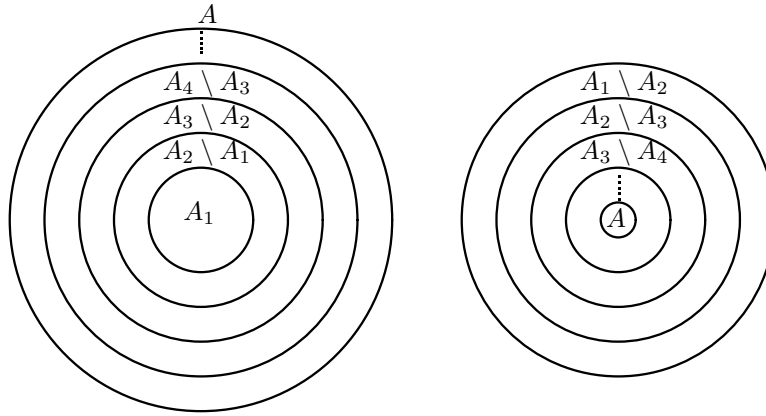


FIGURE 2.2. On the left, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ are nondecreasing concentric disks whose union is the large disk A . On the right, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ are nonincreasing concentric disks whose intersection is the small disk A .

- (2) If μ is a measure, then μ is continuous from above.
(3) If μ is finite-valued; that is, $\mu(A) < \infty$ for each $A \in \mathcal{R}$, then μ is a measure if and only if μ is continuous from above.
(4) If μ is finite-valued, then μ is a measure if and only if μ is continuous from above at \emptyset ; that is, for any nonincreasing sequence of sets $\{A_n\}$ in \mathcal{R} with empty intersection, we have

$$\lim \mu(A_n) = 0.$$

PROOF. To avoid giving a very long proof, we only prove (2) (which implies the “only if” parts of (3) and (4)) and the “only if” part of (1), leaving the rest for your enjoyment.

To prove the “only if” part of (1), assume that μ is a measure. To see that μ is continuous from below, let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ be a nondecreasing sequence of sets in \mathcal{R} with limit set A . Observe that (see the left-hand picture in Figure 2.2)

$$A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup (A_4 \setminus A_3) \cup \dots.$$

The sets on the right are pairwise disjoint, so by countable additivity,

$$\begin{aligned} \mu(A) &= \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1} \setminus A_k) \\ &= \lim_{n \rightarrow \infty} \left(\mu(A_1) + \sum_{k=1}^{n-1} \mu(A_{k+1} \setminus A_k) \right). \end{aligned}$$

Note that $A_n = A_1 \cup \bigcup_{k=1}^{n-1} (A_{k+1} \setminus A_k)$ is a union of pairwise disjoint sets. Thus,

$$\mu(A_n) = \mu(A_1) + \sum_{k=1}^{n-1} \mu(A_{k+1} \setminus A_k),$$

which implies that

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n).$$

To prove (2), assume that μ is a measure and let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a nonincreasing sequence of sets in \mathcal{R} with limit set A such that $\mu(A_1) \neq \infty$. In particular, since $A_n \subseteq A_1$ for each n , by monotonicity we have $\mu(A_n) < \infty$ for all n . Now observe that (see the right-hand picture in Figure 2.2)

$$A_1 = A \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup \cdots .$$

The sets on the right are pairwise disjoint, so by countable additivity and subtractivity of additive set functions, we have

$$\begin{aligned} \mu(A_1) &= \mu(A) + \sum_{k=1}^{\infty} \mu(A_k \setminus A_{k+1}) = \mu(A) + \sum_{k=1}^{\infty} (\mu(A_k) - \mu(A_{k+1})) \\ &= \mu(A) + \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} (\mu(A_k) - \mu(A_{k+1})). \end{aligned}$$

The right-hand sum telescopes:

$$\begin{aligned} \sum_{k=1}^{n-1} (\mu(A_k) - \mu(A_{k+1})) &= (\mu(A_1) - \mu(A_2)) + (\mu(A_2) - \mu(A_3)) \\ &\quad + (\mu(A_3) - \mu(A_4)) + \cdots + (\mu(A_{n-1}) - \mu(A_n)) \\ &= \mu(A_1) - \mu(A_n). \end{aligned}$$

Thus,

$$\mu(A_1) = \mu(A) + \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)).$$

Subtracting $\mu(A_1) < \infty$ from both sides gives

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n),$$

which proves our result. \square

Example 2.2. You may be wondering about the hypothesis “ $\mu(A_1) \neq \infty$ ” in the definition of “continuous from above.” This hypothesis is needed otherwise the result is false. Here’s a trivial counterexample. Consider the set function $\mu : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ defined by $\mu(\emptyset) := 0$ and $\mu(A) := \infty$ for $A \neq \emptyset$. One can check that μ is a measure. Observe that

$$A = \bigcap_{n=1}^{\infty} A_n, \quad \text{where } A = \emptyset \text{ and } A_n = \left(0, \frac{1}{n}\right),$$

and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Thus, $\mu(A) = 0$ and, since $\mu(A_n) = \infty$ for every n , $\lim_{n \rightarrow \infty} \mu(A_n) = \infty$. Hence, $\mu(A) \neq \lim_{n \rightarrow \infty} \mu(A_n)$.

2.1.5. Measures on sequence spaces. Given probability set functions μ_i on $\mathcal{P}(X_i)$ for $i = 1, 2, \dots$, where the X_i ’s are finite sample spaces, we shall prove that the infinite product measure on the cylinder sets of $\prod_{i=1}^{\infty} X_i$,

$$\mu : \mathcal{C} \rightarrow [0, 1],$$

is a measure. In fact, we shall prove even more: In Theorem 2.9 below we shall prove that an arbitrary finitely additive set function on \mathcal{C} is automatically countably additive. We begin with the following lemma.

LEMMA 2.8. *If $\{A_n\}$ is a nonincreasing sequence of nonempty cylinder sets in $\prod_{i=1}^{\infty} X_i$, then $\bigcap_{n=1}^{\infty} A_n$ is not empty.*

PROOF. We prove this lemma in two steps.

Step 1: We first produce a point that should be in the intersection $\bigcap_{n=1}^{\infty} A_n$. To this end, for each n , since each A_n is nonempty we can choose a point in each set; let us denote such points by

$$\begin{aligned} (a_1^1, a_2^1, a_3^1, a_4^1, \dots) &\in A_1 \\ (a_1^2, a_2^2, a_3^2, a_4^2, \dots) &\in A_2 \\ (a_1^3, a_2^3, a_3^3, a_4^3, \dots) &\in A_3 \\ (a_1^4, a_2^4, a_3^4, a_4^4, \dots) &\in A_4 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & \end{aligned}$$

Now look at the first column, which represents a sequence of points in the finite set X_1 . Since X_1 is a finite set, at least one point in X_1 , call such a point a_1 , must be repeated infinitely many times in the first column. Thus, there is an infinite set $B_1 \subseteq \mathbb{N}$ such that $a_1^n = a_1$ for all $n \in B_1$.

Now consider the elements a_2^n in the second column where $n \in B_1$. Since B_1 is infinite, there are infinitely many a_2^n 's in the second column where $n \in B_1$. Since X_2 is a finite set, at least one point in X_2 , call such a point a_2 , is repeated infinitely many times amongst the a_2^n 's where $n \in B_1$. Thus, there is an infinite set $B_2 \subseteq B_1$ such that $a_2^n = a_2$ for all $n \in B_2$. Note that since $B_2 \subseteq B_1$, for all $n \in B_2$ we still have $a_1^n = a_1$. In conclusion, we have an infinite set $B_2 \subseteq B_1$ such that

$$a_1^n = a_1, \quad a_2^n = a_2 \quad \text{for all } n \in B_2.$$

Continuing by induction, we find infinite subsets $B_1, B_2, B_3, \dots \subseteq \mathbb{N}$ and points $a_1, a_2, \dots \in \prod X_i$ such that for each $m \in \mathbb{N}$,

$$(2.2) \quad a_1^n = a_1, \quad a_2^n = a_2, \quad \dots, \quad a_m^n = a_m \quad \text{for all } n \in B_m;$$

we can also choose the sets B_1, B_2, B_3, \dots to be nonincreasing but we won't need this later. We now put

$$a := (a_1, a_2, a_3, \dots) \in \prod X_i.$$

Step 2: We now prove that $a \in A_1 \cap A_2 \cap \dots$. To do so, fix $k \in \mathbb{N}$; we have to prove that $a \in A_k$. By Proposition 1.29 we know that A_k has the form

$$A_k = A \times X_{m+1} \times X_{m+2} \times X_{m+3} \times \dots$$

for some m and for some set $A \subseteq X_1 \times \dots \times X_m$, so we just have to prove that $(a_1, \dots, a_m) \in A$. To prove this, observe that by (2.2), for any $n \in B_m$ we have

$$(a_1, a_2, \dots, a_m, a_{m+1}^n, a_{m+2}^n, \dots) = (a_1^n, a_2^n, \dots, a_m^n, a_{m+1}^n, a_{m+2}^n, \dots),$$

which, as we recall from **Step 1**, is a point belonging to the set A_n . Now, since B_m is an infinite set, we can take $n \geq k$. In this case, we know that $A_n \subseteq A_k$ because $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$, so

$$(a_1, a_2, \dots, a_m, a_{m+1}^n, a_{m+2}^n, \dots) \in A_k.$$

This implies that $(a_1, a_2, \dots, a_m) \in A$, and we're done. □

THEOREM 2.9. *Any finite-valued additive set function on \mathcal{C} is countably additive. In particular, the infinite product of countably many probability set functions is a probability measure on \mathcal{C} .*

PROOF. Let $\mu : \mathcal{C} \rightarrow [0, \infty)$ be a finite-valued additive set function. By Theorem 2.5, to prove that μ is a measure we just have to prove that its extension $\mu : \mathcal{R}(\mathcal{C}) \rightarrow [0, \infty)$ is a measure. Now to prove that this extension is a measure, by Part (4) of Theorem 2.7 we just have to prove that μ is continuous at \emptyset . So, let $\{A_n\}$ be a nonincreasing sequence of sets in $\mathcal{R}(\mathcal{C})$ such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$; we need to show that $\lim_{n \rightarrow \infty} \mu(A_n) = 0$. Now by Lemma 2.8, the assumption $\emptyset = \bigcap_{n=1}^{\infty} A_n$ implies that there is a k such that $A_k = \emptyset$ (otherwise if all the A_n 's were nonempty, their intersection will be nonempty). Since the sets $\{A_n\}$ are nonincreasing we must have

$$A_k = A_{k+1} = A_{k+2} = \cdots = \emptyset.$$

Thus, $\mu(A_n) = 0$ for all $n \geq k$, so we trivially have $\lim_{n \rightarrow \infty} \mu(A_n) = 0$. \square