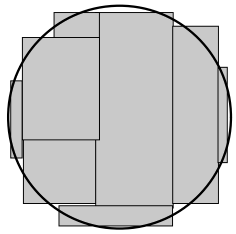

FIGURE 2.3. Here, A is a disk and I is an elementary figure. A is "nearly" equal to I in the sense that the differences $A \setminus I$ and $I \setminus A$ can be made as small as desired

2.4. Approximations and Littlewood's First Principle(s)

John Littlewood (1885–1977).

We know that any measure on a ring $\mathscr{R}$ gives rise to corresponding collection of measurable sets. It is reasonable to consider the elements of $\mathscr{R}$ and countable unions and intersections of such sets as "simple" objects with well-understood properties. Loosely speaking, Littlewood's First Principles, named after John Edensor Littlewood (1885–1977), says a set is measurable if and only if it can be "approximated" arbitrarily close by these simple sets. In $\mathbb{R}^n$ we shall see that a set is Lebesgue measurable if and only if it can be "approximated" arbitrarily close by open sets. Contrast this viewpoint of measurability with the slightly nonintuitive Carathéodory characterization (2.8) of measurability! Thus, Littlewood's First Principles gives an a intuitively satisfying way to think about measurable sets. Littlewood's other principles are discussed in Section **??**.

2.4.1. Littlewood's First Principle(s) for $\mathbb{R}^n$.

Littlewood's First Principle [**48**, p. 26] for subsets of $\mathbb{R}$, which we learned from Royden's classic text [**60**, p. 72], states that

> *Every [finite Lebesgue measurable] set is nearly a finite union of intervals.*

In $\mathbb{R}^n$, we can interpret this principle as follows. Let $A \subseteq \mathbb{R}^n$ have finite Lebesgue outer measure. Then Littlewood's Principle says that A is a Lebesgue measurable set if and only if the set A is "nearly" an elementary figure. We can make the word "nearly" precise: Littlewood's Principle says that A is a Lebesgue measurable set if and only if for any $\varepsilon > 0$, the set A differs from an elementary figure by a set of measure less than ε in the sense that there exists an elementary figure $I \in \mathscr{E}^n$ with (see Figure 2.3)

$$\mathfrak{m}^*(A \setminus I) < \varepsilon \quad \text{and} \quad \mathfrak{m}^*(I \setminus A) < \varepsilon.$$

Thus, the term "nearly" in Littlewood's First Principle can be interpreted as "up to ε-sets" (that is, sets of measure less than ε). In this section we extend this principle to encompass more general measures on $\mathbb{R}^n$ (not just Lebesgue measure) and even for general abstract measure spaces; also, taking advantage of the topological structure of $\mathbb{R}^n$, we can give an alternative formulation of Littlewood's Principles in terms of open and closed sets.

In the following theorem we use the notion of a G_δ **set** (pronounced "gee-delta"), which is a countable intersection of open sets. An F_σ **set** (pronounced "eff-sigma") is a countable union of closed sets. Note that a set is a G_δ set if and only if its complement is an F_σ set. These sets show up often; see Problem 1. With $\mu = \mathfrak{m}$, Lebesgue measure, parts *(2)–(5)* of the following theorem say that Lebesgue measurable sets are "nearly" open sets (or closed sets) and they are "essentially" G_δ sets (or F_σ sets).

THEOREM 2.19 (**Littlewood's First Principle(s) for** $\mathbb{R}^n$). *Let* $A \subseteq \mathbb{R}^n$ *and let* $\mu : \mathscr{E}^n \to [0,\infty)$ *be a measure on the elementary figures* $\mathscr{E}^n$.

(1) If $\mu^*(A) < \infty$, *then* A *is* μ^**-measurable if and only if given* $\varepsilon > 0$ *there is an* $I \in \mathscr{E}^n$ *with*

$$\mu^*(A \setminus I) < \varepsilon \quad \text{and} \quad \mu^*(I \setminus A) < \varepsilon.$$

Without the assumption $\mu^*(A) < \infty$, *the set* A *is* μ^**-measurable if and only if any one of the Properties (2)–(5) hold.*

(2) Given $\varepsilon > 0$ *there is an open set* $\mathcal{U} \subseteq \mathbb{R}^n$ *such that*

$$A \subseteq \mathcal{U} \quad \text{and} \quad \mu^*(\mathcal{U} \setminus A) < \varepsilon.$$

(3) Given $\varepsilon > 0$ *there is a closed set* $C \subseteq \mathbb{R}^n$ *such that*

$$C \subseteq A \quad \text{and} \quad \mu^*(A \setminus C) < \varepsilon.$$

(4) There is a G_δ *set* $G \subseteq \mathbb{R}^n$ *such that*

$$A \subseteq G \quad \text{with} \quad \mu^*(G \setminus A) = 0 \quad \text{and} \quad \mu^*(A) = \mu^*(G).$$

(5) There is an F_σ *set* $F \subseteq \mathbb{R}^n$ *such that*

$$F \subseteq A \quad \text{with} \quad \mu^*(A \setminus F) = 0 \quad \text{and} \quad \mu^*(F) = \mu^*(A).$$

PROOF. We shall prove *(1)*, *(2)*, and *(4)*, leaving the equivalence of measurability to *(3)* and *(5)* for Problem 5.

Step 1: We begin with establishing a useful fact. Let $A \subseteq \mathbb{R}^n$ be arbitrary and let $\varepsilon > 0$. We shall prove there exists an open set $\mathcal{U} \subseteq \mathbb{R}^n$ such that

$$A \subseteq \mathcal{U} \quad \text{and} \quad \mu^*(\mathcal{U}) \le \mu^*(A) + \varepsilon.$$

If $\mu^*(A) = \infty$, then we can take $\mathcal{U} = \mathbb{R}^n$ and we're done, so assume that $\mu^*(A) < \infty$. Then by definition of infimum in the equality

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu(I_k) \ ; \ A \subseteq \bigcup_{k=1}^{\infty} I_k \, , \, I_k \in \mathscr{E}^n \right\}.$$

there are sets $I_k \in \mathscr{E}^n$ such that

$$A \subseteq \bigcup_{k=1}^{\infty} I_k \quad \text{with} \quad \sum_{k=1}^{\infty} \mu(I_k) \le \mu^*(A) + \frac{\varepsilon}{2}. \tag{2.16}$$

For each k, writing $I_k = (a_k, b_k]$, observe that

$$(a_k, b_k] = \bigcap_{j=1}^{\infty} \Big(a_k, b_k + \frac{1}{j}\Big],$$

so by the continuity of measures,

$$\mu(a_k, b_k] = \lim_{j\to\infty} \mu\Big(a_k, b_k + \frac{1}{j}\Big] \quad \Longrightarrow \quad \lim_{j\to\infty} \Big(\mu\Big(a_k, b_k + \frac{1}{j}\Big] - \mu(a_k, b_k]\Big) = 0.$$

Hence, for each k we can choose $\delta_k > 0$ so that

$$\mu(a_k, b_k + \delta_k] - \mu(I_k) < \frac{\varepsilon}{2^{k+1}}.$$

By monotonicity, we have $\mu^*(a_k, b_k + \delta_k) \le \mu^*(a_k, b_k + \delta_k]$ and since μ^* is an extension of μ, we have $\mu^*(a_k, b_k + \delta_k] = \mu(a_k, b_k + \delta_k]$. Thus,

$$\mu^*(a_k, b_k + \delta_k) - \mu(I_k) = \mu(a_k, b_k + \delta_k] - \mu(I_k) < \frac{\varepsilon}{2^{k+1}}.$$

Let $J_k = (a_k, b_k + \delta_k)$ and set $\mathcal{U} = \bigcup_{k=1}^{\infty} J_k$. Then $\mathcal{U}$ is open, $A \subseteq \mathcal{U}$, and

$$\begin{aligned}\mu^*(\mathcal{U}) \le \sum_{k=1}^{\infty} \mu^*(J_k) &\le \sum_{k=1}^{\infty} \Big(\mu(I_k) + \frac{\varepsilon}{2^{k+1}}\Big) \\ &= \sum_{k=1}^{\infty} \mu(I_k) + \frac{\varepsilon}{2} \\ &\le \mu^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu^*(A) + \varepsilon.\end{aligned}$$

Step 2: We now prove that

$$A \text{ is } \mu^*\text{-measurable} \quad \Longrightarrow \quad \textit{(2)} \quad \Longrightarrow \quad \textit{(4)} \quad \Longrightarrow \quad A \text{ is } \mu^*\text{-measurable},$$

which shows the equivalence of measurability to *(2)* and *(4)*.

To prove A is μ^*-measurable $\Longrightarrow$ *(2)*, let $\varepsilon > 0$ and assume A is μ^*-measurable. Writing $\mathbb{R}^n = \bigcup_{k=1}^{\infty} X_k$ where $\{X_k\}$ is a sequence of pairwise disjoint boxes, we have $A = \bigcup_{k=1}^{\infty} A_k$ where $A_k = A \cap X_k$ with $\mu^*(A_k) < \infty$. It follows by **Step 1** that there is an open set B_k with $A_k \subseteq B_k$ and $\mu^*(B_k) - \mu^*(A_k) < \varepsilon/2^k$. This implies that $\mu^*(B_k \setminus A_k) < \varepsilon/2^k$. If $\mathcal{U} = \bigcup_{k=1}^{\infty} B_k$, then $\mathcal{U}$ is open and $\mathcal{U} \setminus A \subseteq \bigcup_{k=1}^{\infty}(B_k \setminus A_k)$, so by countable subadditivity,

$$\mu^*(\mathcal{U} \setminus A) \le \sum_{k=1}^{\infty} \mu^*(B_k \setminus A_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

To see that *(2)* $\Longrightarrow$ *(4)*, assume that for any $\varepsilon > 0$ there is an open set $\mathcal{U}$ such that $A \subseteq \mathcal{U}$ and $\mu^*(\mathcal{U} \setminus A) < \varepsilon$. Then, in particular, for each $k = 1, 2, \ldots$, there is an open set B_k such that $A \subseteq B_k$, and

$$\mu^*(B_k \setminus A) < \frac{1}{k}.$$

Thus, if $B = \bigcap_{k=1}^{\infty} B_k$, then B is a G_δ, $A \subseteq B$, and since $B \subseteq B_k$ for each k, we have

$$\mu^*(B \setminus A) \le \mu^*(B_k \setminus A) < \frac{1}{k}, \quad \text{for all } k \in \mathbb{N}.$$

Since k is arbitrary, it follows that $\mu^*(B \setminus A) = 0$.

Finally, to show that *(4)* $\Longrightarrow$ A is measurable, assume there is a G_δ set G such that

$$A \subseteq G \quad \text{with} \quad \mu^*(G \setminus A) = 0.$$

Then $G \setminus A$ has measure zero so is μ^*-measurable (by Carathéodory's Theorem 2.13). Therefore, $A = G \setminus (G \setminus A)$ is measurable since both G and $G \setminus A$ are.

Step 3: We now prove Property *(1)*. Assume that $\mu^*(A) < \infty$, A is μ^*-measurable, and let $\varepsilon > 0$. Let $B := \bigcup_{k=1}^{\infty} I_k$ where $\bigcup_{k=1}^{\infty} I_k$ appears in (2.16), so that

$$\sum_{k=1}^{\infty} \mu(I_k) \le \mu^*(A) + \frac{\varepsilon}{2} \quad \Longrightarrow \quad \sum_{k=1}^{\infty} \mu(I_k) < \mu^*(A) + \varepsilon.$$

Since the sum $\sum_{k=1}^{\infty} \mu(I_k)$ is finite, there exists an N such that $\sum_{k=N+1}^{\infty} \mu(I_k) < \varepsilon$. Let $I = \bigcup_{k=1}^{N} I_k$. Then $I \in \mathscr{E}^n$ and we shall prove that this set has the required properties. Since $A \subseteq B$, we have $\mu^*(A \setminus I) \leq \mu^*(B \setminus I)$ and since $B \setminus I \subseteq \bigcup_{k=N+1}^{\infty} I_k$, by definition of $\mu^*(B \setminus I)$ we have

$$\mu^*(B \setminus I) \leq \sum_{k=N+1}^{\infty} \mu(I_k) < \varepsilon.$$

Thus, $\mu^*(A \setminus I) < \varepsilon$. To prove that $\mu^*(I \setminus A) < \varepsilon$, observe that since $I \subseteq B$, we have $I \setminus A \subseteq B \setminus A$, so

$$\mu^*(I \setminus A) \leq \mu^*(B \setminus A).$$

Also, $B = \bigcup_{k=1}^{\infty} I_k$, so by definition of $\mu^*(B)$,

$$\mu^*(B) \leq \sum_{k=1}^{\infty} \mu(I_k) < \mu^*(A) + \varepsilon \quad \Longrightarrow \quad \mu^*(B \setminus A) < \varepsilon,$$

where we used that $\mu^*(B \setminus A) = \mu^*(B) - \mu^*(A)$ by subtractivity. This implies that $\mu^*(I \setminus A) < \varepsilon$ and completes the "only if" part of *(1)*.

Step 4: Lastly, we prove the "if" part of *(1)*. So, assume that A, which has finite μ^*-outer measure has the properties in *(1)*; we shall prove that A is μ^*-measurable. Let $\varepsilon > 0$. Then by *(2)* we just have to show there is an open set $\mathcal{U}$ such that $A \subseteq \mathcal{U}$ and $\mu^*(\mathcal{U} \setminus A) < \varepsilon$. To this end, let $\mathcal{U}$ be given by **Step 1** with ε in **Step 1** replaced by $\varepsilon/4$. This implies, in particular, that

$$A \subseteq \mathcal{U} \quad \text{and} \quad \mu^*(\mathcal{U}) < \mu^*(A) + \frac{\varepsilon}{3}.$$

By assumption there is an $I \in \mathscr{E}^n$ such that

$$\mu^*(A \setminus I) < \frac{\varepsilon}{3} \quad \text{and} \quad \mu^*(I \setminus A) < \frac{\varepsilon}{3}.$$

Observe that since $\mathcal{U} = (I \cap \mathcal{U}) \cup (\mathcal{U} \setminus I)$, we have

$$\begin{aligned} \mathcal{U} \setminus A &= \big((I \cap \mathcal{U}) \setminus A\big) \cup \big((\mathcal{U} \setminus I) \setminus A\big) \\ &\subseteq \big(I \setminus A\big) \cup \big(\mathcal{U} \setminus I\big). \end{aligned}$$

Thus,

$$\begin{aligned} \mu^*(\mathcal{U} \setminus A) &\leq \mu^*(I \setminus A) + \mu^*(\mathcal{U} \setminus (I \cap \mathcal{U})) \\ &< \frac{\varepsilon}{3} + \mu^*(\mathcal{U}) - \mu^*(I \cap \mathcal{U}) \\ &< \frac{2\varepsilon}{3} + \mu^*(A) - \mu^*(I \cap \mathcal{U}), \end{aligned} \tag{2.17}$$

where we used that $\mu^*(\mathcal{U}) < \mu^*(A) + \frac{\varepsilon}{3}$. Also observe that

$$A = (I \cap A) \cup (A \setminus I) \subseteq (I \cap \mathcal{U}) \cup (A \setminus I),$$

so

$$\mu^*(A) \leq \mu^*(I \cap \mathcal{U}) + \frac{\varepsilon}{3}.$$

Substituting this into (2.17) we obtain $\mu^*(\mathcal{U} \setminus A) < \varepsilon$. This, finally, completes the proof of Littlewood's First Principles. ☐

To repeat our discussion before this theorem, Littlewood's Principle tells us that μ^*-measurable sets cannot look too horrible, but instead are rather nice: They are "nearly" elementary figures and they are "essentially" Borel sets.

Note that Property *(4)* of Littlewood's Principle looks similar to Property *(2)* of The Regularity Theorem, which states that given an *arbitrary* subset $A \subseteq \mathbb{R}^n$, there is a $B \in \mathscr{B}^n \subseteq \mathscr{M}_{\mu^*}$ such that $A \subseteq B$ and $\mu^*(A) = \mu^*(B)$. However, for an *arbitrary* subset $A \subseteq \mathbb{R}^n$, we may not be able to choose B such that

$$\mu^*(B \setminus A) = 0.$$

Note that *if* there is such a B, *then* it must be the case that A is μ^*-measurable (see the last paragraph in **Step 2** of Theorem 2.19).

Using Property *(5)* of Littlewood's Principles we can completely characterize μ^*-measurable sets.

COROLLARY 2.20. *Let $\mu : \mathscr{E}^n \to [0, \infty)$ be a measure on the elementary figures $\mathscr{E}^n$ of $\mathbb{R}^n$. Then $\mathscr{M}_{\mu^*}$ consists of all disjoint unions*

$$F \cup N,$$

where F is a Borel set and N has measure zero. In fact, we can be even more precise: F can be taken to be an F_σ and N can be taken to be a subset of a G_δ set of μ^*-measure zero.

PROOF. By Theorem 2.19, given $A \in \mathscr{M}_{\mu^*}$ there is an F_σ set $F \subseteq A$ such that $\mu^*(A \setminus F) = 0$. In particular, $A = F \cup N$ is a disjoint union where $N = A \setminus F$ has μ^*-measure zero. Again by the same theorem, N is a subset of a G_δ of the same measure as N, namely zero. Conversely, given a Borel set F and a measure zero set N, the set $F \cup N$ (whether the union is disjoint or not) is measurable since $\mathscr{B}^n = \mathscr{S}(\mathscr{E}^n) \subseteq \mathscr{M}_{\mu^*}$ and the fact that μ^* is complete. $\square$

2.4.2. Regular Borel measures. Notice that as a consequence of **Step 1** in the proof of Theorem 2.19, for any set $A \subseteq \mathbb{R}^n$, we have

$$\mu^*(A) = \inf\{\mu(\mathcal{U})\,;\, A \subseteq \mathcal{U},\ \mathcal{U} \text{ open}\},$$

where we have dropped the "$*$" from $\mu^*(\mathcal{U})$ since $\mathcal{U}$ is open and hence measurable. Thus, we can determine the μ^* outer measure of any subset of $\mathbb{R}^n$ using the open sets. If A is in addition μ^*-measurable, we can also use the compact sets to determine $\mu^*(A)$.

THEOREM 2.21. *Let $\mu : \mathscr{E}^n \to [0, \infty)$ be a measure on the elementary figures of $\mathbb{R}^n$. Then any compact subset of $\mathbb{R}^n$ has finite μ^*-measure and the following regularity properties hold:*

(1) For every set $A \subseteq \mathbb{R}^n$, we have

$$\mu^*(A) = \inf\{\mu(\mathcal{U})\,;\, A \subseteq \mathcal{U},\ \mathcal{U} \textit{ open}\}.$$

(2) For every μ^*-measurable set $A \subseteq \mathbb{R}^n$, we have

$$\mu^*(A) = \sup\{\mu(K)\,;\, K \subseteq A,\ K \textit{ compact}\}.$$

PROOF. As explained above, Property *(1)* follows from **Step 1** in the proof of Theorem 2.19. To prove *(2)*, put

$$S := \{\mu(K)\,;\, K \subseteq A,\ K \text{ compact}\};$$

we need to show that $\mathfrak{m}^*(A) = \sup S$. If $K \subseteq A$, then by monotonicity, $\mu^*(K) \leq \mu^*(A)$, so $\mu^*(A)$ is an upper bound for S. To show that $\mu^*(A)$ is the *least* upper bound of S, let $\alpha < \mu^*(A)$; we shall prove that α is not an upper bound of S. Choose $\varepsilon > 0$ such that $\alpha < \alpha + \varepsilon < \mu^*(A)$.

From Property *(2)* of Theorem 2.19 we know there is a closed set C such that $C \subseteq A$ and $\mu^*(A \setminus C) < \varepsilon$. Since $A = C \cup (A \setminus C)$, we have

$$\mu^*(A) = \mu^*(C) + \mu^*(A \setminus C) \leq \mu^*(C) + \varepsilon.$$

Therefore, $\mu^*(A) \leq \mu^*(C) + \varepsilon$, so

$$\mu^*(C) - \alpha \geq (\mu^*(A) - \varepsilon) - \alpha > 0,$$

recalling that $\alpha + \varepsilon < \mu^*(A)$. Thus,

$$\alpha < \mu^*(C).$$

For each $j \in \mathbb{N}$, let $K_j = C \cap [-j, j]^n$. Then $\{K_j\}$ is a nondecreasing sequence of compact sets such that $C = \bigcup_{j=1}^{\infty} K_j$, so by continuity of measures, we have

$$\mu^*(C) = \lim_{j \to \infty} \mu^*(K_j),$$

By definition of limit and the fact that $\alpha < \mu^*(C)$, it follows that there is some $K = K_j$ such that $\alpha < \mu^*(K)$. Since $K = K_j \subseteq C \subseteq A$, this shows that α is not an upper bound of S. □

In general, given a topological space X such that every compact set is a Borel set, a measure μ on the σ-algebra of Borel subsets of X is said to be a **regular Borel measure** if every compact subset of X has finite μ-measure and for any Borel set $A \subseteq X$, Properties *(1)* and *(2)* of Theorem 2.21 hold with μ^* replaced by μ. Explicitly, a measure $\mu : \mathscr{B}(X) \to [0, \infty]$ on the Borel sets of X is a **regular Borel measure** if for all compact sets $K \subseteq X$, we have $K \in \mathscr{B}(X)$ and $\mu(K) < \infty$, and also

(1) For every Borel set $A \subseteq X$, we have

$$\mu(A) = \inf\{\mu(\mathcal{U})\,;\, A \subseteq \mathcal{U},\ \mathcal{U} \text{ open}\}.$$

(2) For every Borel set $A \subseteq X$, we have

$$\mu(A) = \sup\{\mu(K)\,;\, K \subseteq A,\ K \text{ compact}\}.$$

Thus, Theorem 2.21 implies that given a measure $\mu : \mathscr{E}^n \to [0, \infty)$, the restriction of μ^*, the outer measure induced by μ, to the Borel sets is a regular Borel measure. In particular, when restricted to the Borel sets, Lebesgue measure and Lebesgue-Stieltjes measures are examples of regular Borel measures.

2.4.3. Littlewood's First Principle(s) for the general case. Littlewood's First Principle(s) for general measures says that given a measure μ on a ring $\mathscr{R}$, any set with finite μ^*-outer measure is μ^*-measurable if and only if it is "nearly" an element of $\mathscr{R}$ meaning "up to any ε-set". This is Property *(1)* of the following theorem. We also show that a set is μ^*-measurable if and only if it "essentially" an element of $\mathscr{S}(\mathscr{R})$ meaning "up to a set of measure zero". This is contained in Properties *(2)* and *(3)* of the following theorem. The proof of this theorem is so similar to the proof of Theorem 2.22 that we shall leave its proof to the reader.

THEOREM 2.22 (**Littlewood's First Principle(s) for general case**). *For a measure $\mu : \mathscr{R} \to [0, \infty]$ on a ring $\mathscr{R}$, the following properties hold.*

(1) Let $A \subseteq X$ with $\mu^*(A) < \infty$. Then A is μ^*-measurable if and only if given $\varepsilon > 0$ there is an $I \in \mathscr{R}$ with

$$\mu^*(A \setminus I) < \varepsilon \quad \textit{and} \quad \mu^*(I \setminus A) < \varepsilon.$$

(2) Let $A \subseteq X$ *with* $\mu^*(A) < \infty$. *Then* A *is* μ^**-measurable if and only if there is a* $B \in \mathscr{S}(\mathscr{R})$ *such that*

$$A \subseteq B \quad \text{with} \quad \mu^*(B \setminus A) = 0 \quad \text{and} \quad \mu^*(A) = \mu^*(B).$$

Assume that μ *is* σ*-finite and let* $A \subseteq X$ *(without assuming* $\mu^*(A) < \infty$*). Then* A *is* μ^**-measurable if and only if any one of the following two conditions hold:*

(3) There is a set $B \in \mathscr{S}(\mathscr{R})$ *such that*

$$A \subseteq B \quad \text{with} \quad \mu^*(B \setminus A) = 0 \quad \text{and} \quad \mu^*(A) = \mu^*(B).$$

(4) There is a set $C \in \mathscr{S}(\mathscr{R})$ *such that*

$$C \subseteq A \quad \text{with} \quad \mu^*(A \setminus C) = 0 \quad \text{and} \quad \mu^*(C) = \mu^*(A).$$

We note that Property *(2)* is false in general if we drop the finiteness assumption; see Problem 2. Repeating the proof of Corollary 2.20 gives following characterization of μ^*-measurable sets.

Corollary 2.23. *Let* $\mu : \mathscr{R} \to [0, \infty]$ *be a* σ*-finite measure on a ring. Then* $\mathscr{M}_{\mu^*}$ *consists of all disjoint unions*

$$F \cup N,$$

where $F \in \mathscr{S}(\mathscr{R})$ *and* N *is a subset of an element of* $\mathscr{S}(\mathscr{R})$ *of measure zero. Thus, any* μ^**-measurable set differs from an element of* $\mathscr{S}(\mathscr{R})$ *by a set of measure zero.*