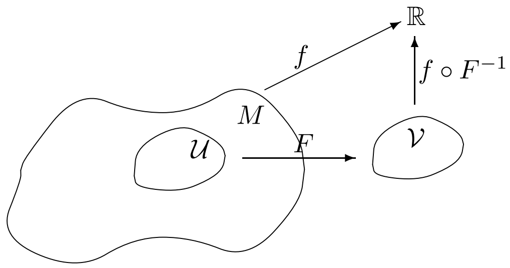

FIGURE 2.8. A function $f : M \to \mathbb{R}$ is smooth if and only if it's smooth on each coordinate patch.

2.2. Smooth functions and partitions of unity

We know what a smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$ is. In this section we generalize smoothness to define smooth functions between manifolds. We also study partitions of unity, an (we can't emphasize enough!) incredibly useful tool in differential geometry.

SECTION OBJECTIVES: THE STUDENT WILL BE ABLE TO ...

- define and give examples of smooth functions between manifolds
- know what a partition of unity is.

2.2.1. Smooth real and complex-valued functions. If $\mathcal{V} \subseteq \mathbb{R}^n$ is open we know what it means for a function $f : \mathcal{V} \to \mathbb{R}$ to be smooth — all partial derivatives of f exist. This is a local condition in the sense that f is smooth on $\mathcal{V}$ if and only if f is smooth (all partial derivatives exist) near (that is, on a neighborhood of) any given point in $\mathcal{V}$. Now let M be an n-dimensional manifold and let $f : M \to \mathbb{R}$ be a function. Then it makes sense to say that f is smooth if and only if it's "locally smooth," which we can take as meaning it's smooth on coordinate patches. More precisely, we say that $f : M \to \mathbb{R}$ is **smooth** or C^∞ if for any coordinate patch[4] $(F, \mathcal{U}, \mathcal{V})$ on M, the function

$$f \circ F^{-1} : \mathcal{V} \to \mathbb{R}$$

is a smooth function on $\mathcal{V}$; see Figure 2.8. Note that $\mathcal{V} \subseteq \mathbb{R}^n$ so we know what it means for $f \circ F^{-1}$ to be smooth.

In a similar way we can also define what it means for a function $f : M \to \mathbb{C}$ to be smooth. The set of real-valued functions on M is denoted by $C^\infty(M, \mathbb{R})$ and the set of all smooth complex-valued functions is denoted by $C^\infty(M)$. Note that $C^\infty(M, \mathbb{R}) \subseteq C^\infty(M)$. We can immediately list some properties of smooth functions. Let's concentrate on $C^\infty(M)$. First, this set of functions is a vector space over $\mathbb{C}$: Given $f, g \in C^\infty(M)$ and $a \in \mathbb{C}$ we define $f + g : M \to \mathbb{C}$ and $af : M \to \mathbb{C}$ in the usual way,

$$(f + g)(p) = f(p) + g(p) \quad , \quad (af)(p) = a \cdot f(p), \quad p \in M.$$

Then $f + g, af \in C^\infty(M)$. We can also multiply functions: $fg : M \to \mathbb{C}$ is defined by

$$(fg)(p) = f(p) \cdot g(p), \quad p \in M.$$

Then $fg \in C^\infty(M)$. See Problem 2 for more on these properties.

[4]Of course, we implicity require the patch to be in the given maximal atlas on M.

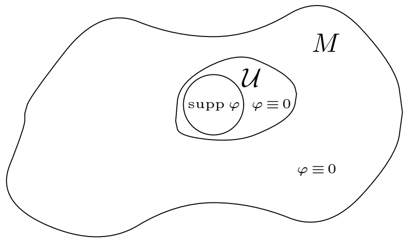

FIGURE 2.9. If $\operatorname{supp}\varphi \subseteq \mathcal{U}$, we can extend φ to be identically 0 outside of $\mathcal{U}$.

We now come to an important question: "Are there any smooth functions?" (Of course, constant functions are smooth but are there others.) This is a legitimate question, for it's not obvious at first sight that nontrivial smooth functions exist! In Theorem 2.4 and Corollary 2.5 below we prove that C^∞ functions do exist and in fact, you can always find and construct them locally. Before going to the proof, we define two notions. First, recall that the **support** of a function $f : M \to \mathbb{R}$ (or $\mathbb{C}$) is defined as

$$\operatorname{supp} f := \overline{\{p \in M \mid f(p) \neq 0\}} = \text{closure of the set of points } \{p \in M \mid f(p) \neq 0\}.$$

By definition of closure, the support of f is the smallest closed set containing the points where f is not zero. In particular, if $p \notin \operatorname{supp} f$, then $f(p) = 0$. The function $f : M \to \mathbb{R}$ is said to be **compactly supported** if $\operatorname{supp} f$ is a compact set. For a manifold M, the set of compactly supported real-valued functions on M is denoted by $C_c^\infty(M, \mathbb{R})$ and for complex-valued functions is denoted by $C_c^\infty(M)$. Note that $C_c^\infty(M, \mathbb{R}) \subseteq C_c^\infty(M)$.

Second, we note that if $\mathcal{U} \subseteq M$ is any open set, then $\mathcal{U}$ is a manifold; we simply take as an atlas those coordinate patches on M that are contained in $\mathcal{U}$. In particular, C^∞ and compactly supported C^∞ (real or complex-valued) functions on $\mathcal{U}$ are defined.

LEMMA 2.3. *Let $\mathcal{U} \subseteq M$ be an open set and let $\varphi \in C_c^\infty(\mathcal{U})$; thus, φ is smooth on $\mathcal{U}$ and $\operatorname{supp}\varphi$ is a compact subset of $\mathcal{U}$. Extend φ to all of M by defining $\varphi \equiv 0$ outside of $\mathcal{U}$ and retain the notation φ for the extended function on M; see Figure 2.9. Then $\varphi \in C^\infty(M)$.*

PROOF. Let $F : \mathcal{W} \to \mathcal{V}$ be a coordinate patch on M. We need to prove that

$$\varphi \circ F^{-1} : \mathcal{V} \to \mathbb{C}$$

is smooth near each point in $\mathcal{V}$. Thus, let us fix $q \in \mathcal{V}$ and prove that $\varphi \circ F^{-1}$ is smooth near q. Let $p \in \mathcal{W}$ with $F(p) = q$. We consider two cases.

Case 1: $p \in \operatorname{supp}\varphi$. Since $\operatorname{supp}\varphi \subseteq \mathcal{U}$ we have $p \in \mathcal{W} \cap \mathcal{U}$. Define

$$G = F|_{\mathcal{W}\cap\mathcal{U}} : \mathcal{W} \cap \mathcal{U} \to F(\mathcal{W} \cap \mathcal{U});$$

then G is a coordinate patch on $\mathcal{U}$. Since φ is a smooth function on $\mathcal{U}$ by assumption, we know that

$$\varphi \circ G^{-1} : F(\mathcal{W} \cap \mathcal{U}) \to \mathbb{C}$$

is smooth. Hence, as $G^{-1} = F^{-1}$ on $F(\mathcal{W} \cap \mathcal{U})$, we see that

$$\varphi \circ F^{-1} \text{ is smooth on the open set } F(\mathcal{W} \cap \mathcal{U}).$$

In particular, since $p \in \mathcal{W} \cap \mathcal{U}$, $q = F(p) \in F(\mathcal{W} \cap \mathcal{U})$, so $\varphi \circ F^{-1}$ is smooth near q.

Case 2: $p \notin \operatorname{supp} \varphi$. Since $\operatorname{supp} \varphi$ is a compact subset of $\mathcal{U}$, it is a compact subset of M[5] and hence K is closed subset of M.[6] Thus, the set $\mathcal{W} \setminus \operatorname{supp} \varphi$ is an open set that contains p. Therefore, since $F^{-1} : \mathcal{V} \to \mathcal{W}$ is a homeomorphism, for x near q, it follows that $F^{-1}(x)$ is in $\mathcal{W} \setminus \operatorname{supp} \varphi$. In particular,

$$\varphi \circ F^{-1}(x) = 0 \text{ for all } x \text{ near } q,$$

so $\varphi \circ F^{-1}$ is certainly smooth near q. □

We can now prove that there exist non-trivial smooth functions on M.

Theorem 2.4. *Given any point $p \in M$ and open set $\mathcal{U}$ containing p, there is a smooth nonnegative function $\varphi : M \to \mathbb{R}$ compactly supported in $\mathcal{U}$ (that is,* $\operatorname{supp} \varphi$ *is compact and* $\operatorname{supp} \varphi \subseteq \mathcal{U}$*) such that $\varphi = 1$ on a neighborhood of p.*

Proof. By Proposition 2.2 there is a coordinate patch $F : \mathcal{W} \to \mathbb{R}^n$ with $p \in \mathcal{W}$, $\mathcal{W} \subseteq \mathcal{U}$, and $F(p) = 0$. As in Example 2.4 we can define a smooth nonnegative function $\psi : \mathbb{R}^n \to \mathbb{R}$ that vanishes outside the ball $\{x \in \mathbb{R}^n \mid \|x\| \leq 2\}$ and is 1 on the ball $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ (just choose $a = 2$ and $\varepsilon = 1$ in that example). We leave you the pleasure of proving that

$$\varphi := \psi \circ F : \mathcal{W} \to \mathbb{R}$$

is in $C_c^\infty(\mathcal{W}, \mathbb{R})$, is nonnegative, and is identically 1 in neighborhood of p. By the previous lemma we can extend φ to be identically zero outside of $\mathcal{W}$ and still have a smooth function; this function has all the properties we need. □

Theorem 2.4 is useful in defining partitions of unity, our next subject, but first:

Corollary 2.5. *Let $\mathcal{U} \subseteq M$ be open, let $p \in \mathcal{U}$, and let $f \in C^\infty(\mathcal{U})$. Then there is a function $g \in C^\infty(M)$ such that $g \equiv f$ on a neighborhood of p.*

Proof. By Theorem 2.4, there is a smooth function $\varphi : M \to \mathbb{R}$ compactly supported in $\mathcal{U}$ such that $\varphi = 1$ on a neighborhood of p. It follows that

$$g := \varphi \cdot f,$$

the product of φ and f, is a smooth function on $\mathcal{U}$ with compact support. Hence by Lemma 2.3 we can extend g to be zero outside of $\mathcal{U}$ to get a smooth function g on M. Note that $g \equiv f$ near p because $\varphi \equiv 1$ near p. □

This theorem shows that there are *many* smooth functions on M because we can always "extend" functions defined on coordinate patches to all of M! Explicitly, let $F : \mathcal{U} \to \mathcal{V}$ be a coordinate patch on M and let $h : \mathcal{V} \to \mathbb{R}$ be any smooth function you can conjure up. Then $f := h \circ F$ is a smooth function on $\mathcal{U}$ (as is easily checked) so by Corollary 2.4 there is a smooth function g on M that equals f on a neighborhood in $\mathcal{U}$.

---

[5]Any cover of K by open sets in M induces a cover of K by open sets in $\mathcal{U}$ by intersecting the cover with $\mathcal{U}$. Since $K \subseteq \mathcal{U}$ is compact, it can be covered by finitely many sets in the cover intersected with $\mathcal{U}$, and hence with finitely many sets in the original cover.

[6]This is because any compact subset of a Hausdorff space is closed (and manifolds are by assumption Hausdorff). In the sequel will sometimes omit "elementary" topological facts that we mentioned in this footnote and the previous one, so be careful!

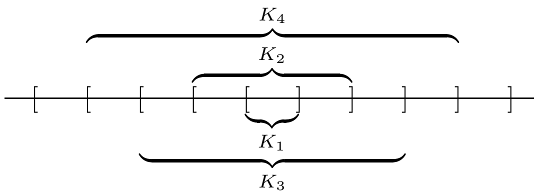

FIGURE 2.10. The compact sets K_j grow "nicely". (Two or three-dimensional examples could be a sequence of concentric compact discs or balls getting bigger and bigger.)

2.2.2. Partitions of unity. A **partition of unity** on a manifold M is a collection of countably many smooth real-valued functions $\{\varphi_k\}$ on M such that

(1) $0 \le \varphi_k \le 1$ for each k.
(2) Any given compact subset of M intersects only finitely many supports of the φ_k. (We say that the supports $\{\operatorname{supp}\varphi_k\}$ are locally finite.)
(3) $\sum_{k=1}^{\infty} \varphi_k(x) = 1$ for each $x \in M$.

By Property (2), given any compact set $K \subseteq M$, there is an N such that

$$\sum_{k=1}^{\infty} \varphi_k(x) = \varphi_1(x) + \cdots + \varphi_N(x) \quad \text{for all } x \in K,$$

so the sum $\sum_{k=1}^{\infty} \varphi_k(x)$ is really only a finite sum on any given compact set. Let $\{\mathcal{U}_\alpha\}$ be a cover of M by open sets. The partition of unity $\{\varphi_k\}$ is said to be **subordinate to the cover** $\{\mathcal{U}_\alpha\}$ if for each k, the support of φ_k lies in some $\mathcal{U}_\alpha$. A convention is that if we say $\{\varphi_\alpha\}$ is a "partition of unity subordinate to the cover $\{\mathcal{U}_\alpha\}$" using the *same* index α for the functions as the sets, we mean that only countably many of the φ_α's are not identically zero, Properties (1)–(3) above hold, and that each φ_α is supported in $\mathcal{U}_\alpha$.

Theorem 2.7 below on the existence of partitions of unity is one of the most useful theorems in this book, but the proof is (unfortunately) long and detailed. We start off with the following lemma.

LEMMA 2.6. *For any manifold M there are countably many compact sets $\{K_j\}$ such that*

(i) M is covered by $\{K_j\}$ and $\{\mathring{K}_j\}$, where "$\circ$" means interior.
(ii) The K_j's have the property that they grow as follows (see Figure 2.10):

$$K_1 \subseteq \mathring{K}_2 \subseteq K_2 \subseteq \mathring{K}_3 \subseteq K_3 \subseteq \mathring{K}_4 \subseteq K_4 \subseteq \cdots.$$

*The collection $\{K_j\}$ is referred to as a **compact exhaustion** of M.*

PROOF. We prove this lemma in four steps.

Step 1: We first prove this lemma for $\mathbb{R}^n$, which is easy: Just let

$$K_j := B_j(0) = \text{ the ball of radius } j \text{ centered at } 0.$$

The rest of the proof consists of writing M as a union of coordinate patches, each of which is homeomorphic to $\mathbb{R}^n$ and applying *Step 1* ... hold on to your seats!

Step 2: We now prove that any open subset of $\mathbb{R}^n$ can be covered by countably many open balls. Indeed, observe that since the rational numbers are countable, we can list the points in $\mathbb{R}^n$ with rational coordinates: $\{p_1, p_2, \ldots\}$ and we can list the positive rational numbers: $\{r_1, r_2, \ldots\}$. In particular, the collection of open balls $B_{r_i}(p_j) = \{x \in \mathbb{R}^n \mid \|x - p_j\| < r_i\}$ is countable. Thus, the subcollection of open balls

$$\{\, B_{r_i}(p_j) \mid B_{r_i}(p_j) \subseteq \mathcal{V}\,\} \tag{2.5}$$

is also countable. We claim that this subcollection covers $\mathcal{V}$. To see this, let $p \in \mathcal{V}$. Since $\mathcal{V} \subseteq \mathbb{R}^n$ is open we can choose a rational number r_i such that

$$B_{2r_i}(p) \subseteq \mathcal{V}. \tag{2.6}$$

Since the points in $\mathcal{V}$ with rational coordinates is dense in $\mathcal{V}$ (because the rational numbers are dense in $\mathbb{R}$) we can find a point $p_j \in \mathcal{V}$ such that $\|p - p_j\| < r_i$. We shall prove that $B_{r_i}(p_j) \subseteq \mathcal{V}$. To this end, let $x \in B_{r_i}(p_j)$. Then by the triangle inequality,

$$\|x - p\| = \|x - p_j + p_j - p\| \leq \|x - p_j\| + \|p_j - p\| < r_i + r_i = 2r_i.$$

Hence, by (2.6), we see that $x \in \mathcal{V}$. Therefore, the collection (2.5) does cover $\mathcal{V}$.

Step 3: Next, we prove that M can be covered by countably many coordinate patches (of course, in the given maximal atlas) each of which is homeomorphic to $\mathbb{R}^n$. Indeed, by definition of manifold we can always cover M by countably many coordinate patches. Since a countable union of countable sets is countable, we are thus reduced to proving that if $(F, \mathcal{U}, \mathcal{V})$ is a coordinate patch on M where $\mathcal{V} \subseteq \mathbb{R}^n$ is open, then $\mathcal{U}$ can be covered by countably many coordinate patches $\{(F_i, \mathcal{U}_i, \mathcal{V}_i)\}$ where $\mathcal{V}_i = \mathbb{R}^n$ for all i. To see this, by *Step 2*, we know that $\mathcal{V}$ is a countable union of open balls $\{B_i\}$. In particular, if $\mathcal{U}_i := F^{-1}(B_i)$, then $\{\mathcal{U}_i\}$ covers $\mathcal{U}$. For each i, choose a diffeomorphism $G_i : B_i \to \mathbb{R}^n$ (see Example 2.6) and define

$$F_i : \mathcal{U}_i \to \mathbb{R}^n \quad \text{by} \quad F_i := G_i \circ F : \mathcal{U}_i \to \mathbb{R}^n.$$

Then it's easy to check that $\{(F_i, \mathcal{U}_i, \mathbb{R}^n)\}$ is a countable collection of coordinate patches covering $\mathcal{U}$.

Step 3: We now prove our result. To do so, apply *Step 3* to cover M by countably many coordinate patches $\{(F_i, \mathcal{U}_i, \mathcal{V}_i)\}$ where $\mathcal{V}_i = \mathbb{R}^n$ for all i. *Step 1* holds for each $\mathcal{V}_i$ and since $F_i : \mathcal{U}_i \to \mathcal{V}_i$ is a homeomorphism, it also holds for each $\mathcal{U}_i$. Hence, for each i, we can find a cover $\{K_{ij}\}$ of $\mathcal{U}_i$ by compact subsets of $\mathcal{U}_i$ such that

(i) $\mathcal{U}_i$ is covered by $\{K_{ij}\}$ and $\{\overset{\circ}{K}_{ij}\}$.
(ii) The K_{ij}'s have the property that they grow as follows:

$$K_{i1} \subseteq \overset{\circ}{K}_{i2} \subseteq K_{i2} \subseteq \overset{\circ}{K}_{i3} \subseteq K_{i3} \subseteq \overset{\circ}{K}_{i4} \subseteq K_{i4} \subseteq \cdots.$$

Now for each k, defining

$$K_k := \bigcup_{i,j \leq k} K_{ij},$$

you can check that we get the cover $\{K_k\}$ of M that we want. □

Here is our incredibly useful theorem.

THEOREM 2.7. *Let $\{\mathcal{U}_\alpha\}$ be a cover of M by open sets. Then there exists a partition of unity $\{\varphi_k\}$ subordinate to the cover $\{\mathcal{U}_\alpha\}$ such that* $\operatorname{supp} \varphi_k$ *is compact for each k. If we do not require the supports of each φ_k to be compact, we can always find a partition of unity $\{\varphi_\alpha\}$ subordinate to the cover $\{\mathcal{U}_\alpha\}$. Finally, if M is compact, the partition of unity can be chosen finite in number.*

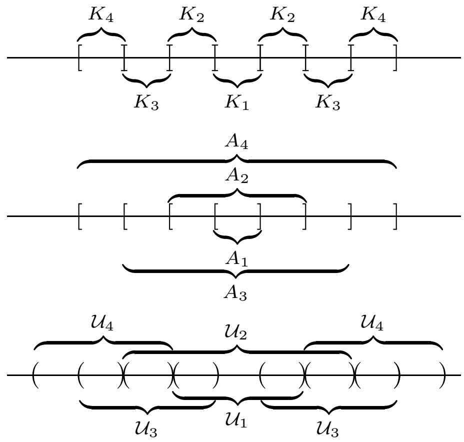

FIGURE 2.11. Pictorial of the sets $\{K_j\}$ and $\{\mathcal{U}_j\}$.

PROOF. We prove this theorem in three steps.

Step 1: The first thing to do is to construct countable covers $\{K_j\}$ and $\{\mathcal{U}_j\}$ of M having "nice" properties:

(i) K_j is compact and $\mathcal{U}_j$ is open.
(ii) $K_j \subseteq \mathcal{U}_j$.
(iii) Any given compact subset of M intersects at most finitely many of the $\mathcal{U}_j$'s.

To prove this, let $\{A_k\}$ be a compact exhaustion of our manifold proved in our lemma. Define

$$\begin{array}{rcl} K_1 := A_1 & \subseteq & \mathcal{U}_1 := \mathring{A}_2, \\ K_2 := A_2 \setminus \mathring{A}_1 & \subseteq & \mathcal{U}_2 := \mathring{A}_3 \\ K_3 := A_3 \setminus \mathring{A}_2 & \subseteq & \mathcal{U}_3 := \mathring{A}_4 \setminus A_1 \\ K_4 := A_4 \setminus \mathring{A}_3 & \subseteq & \mathcal{U}_4 := \mathring{A}_5 \setminus A_2 \\ K_5 := A_5 \setminus \mathring{A}_4 & \subseteq & \mathcal{U}_5 := \mathring{A}_6 \setminus A_3 \\ \vdots & \vdots & \vdots \end{array}$$

See Figure 2.11 to see how the K_j's and $\mathcal{U}_j$'s are formed. By definition, each K_j is compact and $\mathcal{U}_j$ is open, and $K_j \subseteq \mathcal{U}_j$. To see that $\{K_j\}$ and $\{\mathcal{U}_j\}$ cover M we just have to prove that $\{K_j\}$ covers M. To this end, let $p \in M$. Since $\{A_j\}$ covers M, we can choose the smallest j such that $p \in A_j$. Then $p \in A_j \setminus A_{j-1}$ and hence $p \in A_j \setminus \mathring{A}_{j-1} = K_j$. Thus, $\{K_j\}$ covers M. To verify Property (iii), let $K \subseteq M$ be compact. Since $\{\mathring{A}_k\}$ is a cover of M by open sets and K is compact, there is an N such that

$$K \subseteq \mathring{A}_1 \cup \cdots \cup \mathring{A}_N .$$

Now the property $A_1 \subseteq \mathring{A}_2 \subseteq A_2 \subseteq \mathring{A}_3 \subseteq A_3 \subseteq \mathring{A}_4 \subseteq A_4 \subseteq \cdots$ implies, in particular, that $K \subseteq A_N$ and that $A_N \subseteq A_{j-2}$ for $j \geq N+2$. Hence, for $j \geq N+2$, we have

$$K \cap \mathcal{U}_j \subseteq A_N \cap \mathcal{U}_j = A_N \cap (\mathring{A}_{j+1} \setminus A_{j-2}) = \varnothing.$$

This completes *Step 1.*

Step 2: Let $K \subseteq \mathcal{U} \subseteq M$ where K is compact and $\mathcal{U}$ is open. We show there exists finitely many smooth nonnegative functions ψ_k on M all of which are compactly supported in $\mathcal{U}$ such that

(a) The support of ψ_k is contained in some $\mathcal{U}_\alpha$.
(b) $\sum_k \psi_k(p) \geq 1$ for all $p \in K$.

To see this, note that because $\{\mathcal{U}_\alpha\}$ covers M, given $p \in K$ there is an open set $\mathcal{U}_\alpha$ with $p \in \mathcal{U}_\alpha$. Since $K \subseteq \mathcal{U}$, $p \in \mathcal{U} \cap \mathcal{U}_\alpha$, so by Theorem 2.4 there is a smooth nonnegative function ψ_p on M equal to one on a neighborhood $\mathcal{V}_p \subseteq \mathcal{U} \cap \mathcal{U}_\alpha$ of p. Doing this for each $p \in K$ we get a cover $\{\mathcal{V}_p\}_{p \in K}$ of K by open sets. Since K is compact, it is covered by finitely many $\mathcal{V}_p$'s, say $\mathcal{V}_{p_1}, \ldots, \mathcal{V}_{p_N}$. Let $\psi_k := \psi_{p_k}$. Then by construction, the support of ψ_k is contained in some $\mathcal{U}_\alpha$. If $p \in K$, then $p \in \mathcal{V}_{p_k}$ for some k, in which case $\psi_k(p) = 1$. Hence, $\sum_k \psi_k(p) \geq 1$ on K.

Step 3: We now finish the proof. Let $\{K_j\}$ and $\{\mathcal{U}_j\}$ be the sets constructed in *Step 1*. Applying *Step 2* to the sets $K_j \subseteq \mathcal{U}_j$, we see that for each j there exists finitely many smooth nonnegative functions $\psi_{j1}, \psi_{j2}, \psi_{j3}, \ldots$ (a finite list) on M compactly supported in $\mathcal{U}_j$ such that

(I) The support of ψ_{jk} is contained in some $\mathcal{U}_\alpha$.
(II) $\sum_k \psi_{jk}(p) \geq 1$ for all $p \in K_j$.

Note that since any given compact subset of M intersects only finitely many of the $\mathcal{U}_j$'s (Property (iii) in *Step 1*) and $\operatorname{supp} \psi_{jk} \subseteq \mathcal{U}_j$, a compact subset of M will intersect only finitely many supports of the ψ_{jk}'s. In particular, the function $\psi = \sum_{jk} \psi_{jk}$ is really only a finite sum on any given compact set, so defines a smooth function on M. Note that $\psi \geq \psi_{jk}$ for any j, k and by Property (II) and the fact that $\{K_j\}$ covers M, we have $\psi \geq 1$ everywhere on M. For each j, k, define

$$\varphi_{jk} := \frac{\psi_{jk}}{\psi}.$$

Then φ_{jk} is compactly supported in some $\mathcal{U}_\alpha$ since ψ_{jk} was, $0 \leq \varphi_{jk} \leq 1$ since ψ_{jk} is nonnegative and $\psi \geq \psi_{jk}$, and

$$\sum_{jk} \varphi_{jk} = \sum_{jk} \frac{\psi_{jk}}{\psi} = \frac{1}{\psi} \sum_{jk} \psi_{jk} = \frac{\psi}{\psi} = 1.$$

The set $\{\varphi_{jk}\}$ gives the partition of unity we were after.

To prove that there is a partition of unity $\{\varphi_\alpha\}$ subordinate to the cover $\{\mathcal{U}_\alpha\}$, let us first re-index the partition of unity just found with a single index "k" instead of two indices "jk", so let's use the notation $\{\varphi_k\}$. Then for each k choose (and fix) an α_k such that $\operatorname{supp} \varphi_k \subseteq \mathcal{U}_{\alpha_k}$. Now for each α, define

$$\psi_\alpha := \begin{cases} 0 & \text{if there is no } k \text{ such that } \alpha = \alpha_k\text{; otherwise,} \\ \displaystyle\sum_{k \,|\, \alpha_k = \alpha} \varphi_k & \text{where the sum is only over those } k \text{ such that } \alpha_k = \alpha. \end{cases}$$

Note that $\psi_\alpha = 0$ unless $\alpha = \alpha_k$ for some k, hence only countably many of the ψ_α's are not identically zero. Also note that

$$\operatorname{supp} \psi_\alpha \subseteq \bigcup_{k \,|\, \alpha_k = \alpha} \operatorname{supp} \varphi_k. \tag{2.7}$$

For each k on the right-hand side, we have $\operatorname{supp} \varphi_k \subseteq \mathcal{U}_\alpha$ by definition of α_k, so $\operatorname{supp} \psi_\alpha \subseteq \mathcal{U}_\alpha$. Finally, let us verify Properties (1)–(3) of being a partition of unity.

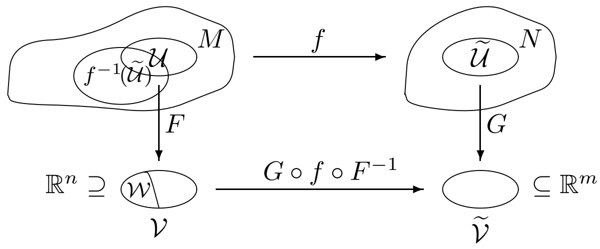

FIGURE 2.12. A function $f : M \to N$ is smooth if and only if it's smooth on coordinate patches.

First, since $\sum \varphi_k = 1$, we have $0 \le \psi_\alpha \le 1$. Second, to prove local finiteness for $\{\operatorname{supp} \psi_\alpha\}$, let $K \subseteq M$ be compact. Then by local finiteness for $\{\varphi_k\}$ there is an ℓ such that $\operatorname{supp} \varphi_k \cap K = \varnothing$ for $k > \ell$. In particular, if $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_\ell$, then $\operatorname{supp} \psi_\alpha \cap K = \varnothing$ because only $k > \ell$ can appear on the right in (2.7). This proves the local finiteness for $\{\psi_\alpha\}$. Lastly, the third property is easy:

$$\sum_\alpha \psi_\alpha = \sum_\alpha \sum_{\alpha_k = \alpha} \varphi_k = \sum_k \varphi_k = 1.$$

Finally, if M is compact, the partition of unity can be chosen finite in number because we can simply choose a finite subcover of $\{\mathcal{U}_\alpha\}$ at the beginning and find a partition of unity with respect to the finite subcover. ☐

2.2.3. Smooth functions between manifolds. If $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{V} \subseteq \mathbb{R}^n$, then we know what it means for a function $f : \mathcal{U} \to \mathcal{V}$ to be smooth (all coordinate functions of f are infinitely differentiable.) Now let M be an m-dimensional manifold and N be an n-dimensional manifold and let $f : M \to N$ be a function. Since M and N locally look like $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, it makes sense to call f smooth if it is smooth in coordinate patches. More precisely, let $f : M \to N$ be continuous and let $(F, \mathcal{U}, \mathcal{V})$ and $(G, \widetilde{\mathcal{U}}, \widetilde{\mathcal{V}})$ be coordinate patches in M and N, respectively. Since f is continuous, the set $f^{-1}(\widetilde{\mathcal{U}}) \subseteq M$ is open and therefore the set $\mathcal{W} := F(f^{-1}(\widetilde{\mathcal{U}}) \cap \mathcal{U}) \subseteq \mathbb{R}^m$ is open. We say that f is **smooth** or C^∞ if the function

$$G \circ f \circ F^{-1} : \mathcal{W} \to \widetilde{\mathcal{V}}$$

is smooth for all such coordinate patches. See Figure 2.12 for a picture of this situation. We say that $f : M \to N$ is a **diffeomorphism** if f is a homeomorphism, f is smooth, and $f^{-1} : N \to M$ is also smooth. In this case, we say that M and N are **diffeomorphic**.

Example 2.13. (See Problem 3 for more examples like this one.) Let $M = \mathbb{R}$ as a *set*. Consider the function $F : M \to \mathbb{R}$ defined by $F(x) := x^3$. Then F is a bijection so it defines a coordinate patch on M. Putting the topology on M induced by the atlas $\{F\}$, by Proposition 2.1 we know that $F : M \to \mathbb{R}$ is a homeomorphism. In particular, M is Hausdorff. Therefore, completing the atlas $\{F\}$ we get a manifold, which we shall denote by $\mathbb{R}_3$. Here's a nonintuitive fact: The identity map $\operatorname{Id} : \mathbb{R}_3 \to \mathbb{R}$ (defined by $\operatorname{Id}(x) = x$) is not smooth! Indeed, by definition of smoothness, $\operatorname{Id} : \mathbb{R}_3 \to \mathbb{R}$ is smooth if and only if

$$\operatorname{Id} \circ F^{-1} : \mathbb{R} \to \mathbb{R}$$

is smooth. However, $F^{-1}(x) = x^{1/3}$ so $\mathrm{Id} \circ F^{-1}(x) = \mathrm{Id}(x^{1/3}) = x^{1/3}$. Since $x^{1/3}$ fails to be differentiable at 0, it follows that Id is not smooth! However, even though Id is not smooth it does turn out that $\mathbb{R}_3$ is diffeomorphic to $\mathbb{R}$; the coordinate patch F furnishes the diffeomorphism as the following proposition shows.

Proposition 2.8. *Let $(F, \mathcal{U}, \mathcal{V})$ be a coordinate patch on a manifold M. Then $F : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism from $\mathcal{U} \subseteq M$ onto $\mathcal{V} \subseteq \mathbb{R}^n$. Simply put, coordinate patches are diffeomorphisms.*

Proof. Recall that $\mathcal{U} \subseteq M$ is a manifold with atlas taken as those coordinate patches on M that are contained in $\mathcal{U}$. We just have to prove that $F : \mathcal{U} \to \mathcal{V}$ is smooth and $F^{-1} : \mathcal{V} \to \mathcal{U}$ is also smooth. To prove that $F : \mathcal{U} \to \mathcal{V}$ is smooth, let $G : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{V}}$ be a coordinate patch on $\mathcal{U}$; we need to show that

$$F \circ G^{-1} : \widetilde{\mathcal{V}} \to \mathcal{V}$$

is smooth. However, this is smooth by definition of compatibility (being coordinate patches in the atlas)! Similarly, one can show by a "definition argument" that $F^{-1} : \mathcal{V} \to \mathcal{U}$ is also smooth. □

Exercises 2.2.

1. Prove that $C^\infty(M, \mathbb{R})$ "separates points" of M in the sense that given any two distinct points $p, q \in M$ there is a function $f \in C^\infty(M, \mathbb{R})$ such that $f \equiv 1$ on a neighborhood of p and $f \equiv 0$ on a neighborhood of q.
2. Here are some properties that we'll use without thinking:
   (i) Prove that the composition of smooth maps is smooth. That is, if $g : M_1 \to M_2$ and $f : M_2 \to M_3$ are smooth maps between manifolds, then $f \circ g : M_1 \to M_3$ is smooth.
   (ii) An **algebra** over a field $\mathbb{F}$ is a vector space V over $\mathbb{F}$ that is also a commutative ring such that scalar multiplication respects multiplication in the sense that for any $a \in \mathbb{F}$ and $f, g \in V$, we have $a \cdot (f \cdot g) = (af) \cdot g = f \cdot (ag)$. Prove that $C^\infty(M, \mathbb{R})$ is an algebra over $\mathbb{R}$ and $C^\infty(M)$ is an algebra over $\mathbb{C}$.
3. Let $n \in \mathbb{N}$ be odd and let $M = \mathbb{R}$ as a set. Consider the function $F_n : M \to \mathbb{R}$ defined by $F_n(x) := x^n$. Then F_n is a bijection so defines a coordinate patch on M.
   (i) Prove that completing the atlas $\{F_n\}$ we get a manifold, which we shall denote by $\mathbb{R}_n$. We now analyze some properties of $\mathbb{R}_n$.
   (ii) Show that a function $f : \mathbb{R}_n \to \mathbb{R}_m$ is smooth if and only if the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x^{m/n})$ is smooth in the usual sense. In particular, the identity map $i : \mathbb{R}_n \to \mathbb{R}_m$ is smooth if and only if $m/n \in \mathbb{N}$.
   (iii) For $n, m \in \mathbb{N}$ odd, prove that the identity map $i : \mathbb{R}_n \to \mathbb{R}_m$ is a diffeomorphism if and only if $m = n$.
   (iv) Nonetheless, prove that for any $n, m \in \mathbb{N}$ odd, $\mathbb{R}_n$ is diffeomorphic to $\mathbb{R}_m$.
4. In this problem we need Problems 1 and 2 in Exercises 2.1. Let $n \in \mathbb{N}$ and consider the projection map $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ defined by

$$\pi(p) := [p],$$

   where $[p]$ denotes the equivalence class of $p \in \mathbb{S}^n$ in $\mathbb{R}P^n$. Here are some problems in increasing order of difficultly:
   (i) Prove that π is smooth.
   (ii) Prove that a function $f : \mathbb{R}P^n \to \mathbb{R}$ is smooth if and only if $f \circ \pi : \mathbb{S}^n \to \mathbb{R}$ is smooth.
   (iii) Let M be a manifold. Prove that a function $f : M \to \mathbb{S}^n$ is smooth if and only if f is continuous and $\pi \circ f : M \to \mathbb{R}P^n$ is smooth. Can you remove the continuity condition?

5. Let A and $\mathcal{U}$ be closed, respectively, open, subsets of M with $A \subseteq \mathcal{U}$. Prove that there is a function $f \in C^\infty(M, \mathbb{R})$ such that $0 \leq f \leq 1$, $f = 1$ on A, and $f = 0$ outside of $\mathcal{U}$. This is a smooth version of the famous **Urysohn lemma** of topology fame.

2.3. Tangent and cotangent vectors

In some sense the only inherent objects on an abstract manifold are the smooth functions and the goal of differential geometry is to analyze manifolds by using these functions to define geometric objects (e.g. vectors) that can be used to better understand the manifolds. In Euclidean space a vector is an arrow that indicates both magnitude and direction. In abstract manifolds, such "arrows" do not make sense. The goal of this section is to understand vectors on manifolds in terms of the only thing manifolds have at this point: the smooth functions.

Section objectives: The student will be able to ...

- state the defining properties of a tangent vector on a manifold.
- explain local coordinate notation.
- know what a cotangent vector is.

2.3.1. Review of vectors in Euclidean space. Taking $\mathbb{R}^2$ for example, from calculus we know that a vector v at a point $p \in \mathbb{R}^2$ is simply an arrow

$$v = a\,\vec{i} + b\,\vec{j},$$

which represents the "directed line segment" emanating from the point p to the point $p + (a, b)$. In order to transfer this concept to a manifold we need a notion of vector in terms of smooth functions; this, fortunately, has already been done for us through the concept of directional derivative. Recall that if f is a smooth (say real-valued) function defined near p, the **directional derivative** of f in the direction of v is by definition the real number

$$D_v f := \frac{d}{dt}\bigg|_{t=0} f(p + tv).$$

This is interpreted as a type of rate of change of f with respect to v. We only need f to be defined near p because the derivative only depends on the values of f close to p. Thus, D_v takes a smooth function f defined near p and gives a real number $D_v f$. With this in mind, define

$$C_p := \{\text{real-valued functions that are defined and smooth near } p\}.$$

In other words, $f \in C_p$ if and only if $f : \mathrm{Dom}(f) \to \mathbb{R}$ where $\mathrm{Dom}(f) \subseteq \mathbb{R}^2$ and there is an open set $\mathcal{U} \subseteq \mathrm{Dom}(f)$ containing p such that $f : \mathcal{U} \to \mathbb{R}$ is smooth. Notice that C_p is a vector space. For example, let $f : \mathrm{Dom}(f) \to \mathbb{R}$ and $g : \mathrm{Dom}(g) \to \mathbb{R}$ be elements of C_p. Then there is an open set $\mathcal{U} \subseteq \mathrm{Dom}(f)$ containing p such that $f : \mathcal{U} \to \mathbb{R}$ is smooth and there is an open set $\mathcal{V} \subseteq \mathrm{Dom}(g)$ containing p such that $g : \mathcal{V} \to \mathbb{R}$ is smooth. Then it follows that

$$f + g : \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \to \mathbb{R}$$

has the property that $\mathcal{U} \cap \mathcal{V} \subseteq \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ is open, contains p, and

$$f + g : \mathcal{U} \cap \mathcal{V} \to \mathbb{R}$$