

(ii) Let  $\varphi : V_1 \times \cdots \times V_k \rightarrow \text{Fr}(V_1 \times \cdots \times V_k)/I$  be the quotient map and define  $V_1 \otimes \cdots \otimes V_k := \text{Fr}(V_1 \times \cdots \times V_k)/I$ . Prove that  $(V_1 \otimes \cdots \otimes V_k, \varphi)$  has the universal mapping property for multi-linear maps on  $V_1 \times \cdots \times V_k$ .

In this sense, the tensor product  $V_1 \otimes \cdots \otimes V_k$  “solves” the **universal mapping problem** for multi-linear maps on  $V_1 \times \cdots \times V_k$ .

## 2.6. The exterior algebra

We study the exterior algebra and the all-important wedge product.

SECTION OBJECTIVES: THE STUDENT WILL BE ABLE TO ...

- explain what a  $k$ -form is.
- define the wedge product and state its properties.
- Define the determinant of a linear map using the exterior algebra.
- describe the contraction operator.

**2.6.1. Permutations, pullbacks, and  $k$ -forms.** We continue our study of multi-linear algebra in the spirit of our pedestrian (= ordinary, nothing special, common ...) approach based on familiar notions from linear algebra; see Problem 2 for a more sophisticated approach to the exterior algebra.

From the “prelude to tensors” section we already know what an alternating tensor is, but here is a formal definition. Let  $V$  be a finite-dimensional  $\mathbb{K}$  (=  $\mathbb{R}$  or  $\mathbb{C}$ ) vector space and let  $\alpha \in (V^*)^{\otimes k}$ , which means that

$$\alpha : V^k \rightarrow \mathbb{K}$$

is multi-linear. The tensor  $\alpha \in (V^*)^{\otimes k}$  is called **alternating, anti-symmetric**, or an **(alternating)  $k$ -form**, if for all  $(v_1, \dots, v_k) \in V^k$  and  $i < j$ , we have

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Thus,  $\alpha$  changes sign whenever two of its arguments are switched. Note that the sum of any two  $k$ -forms and the product of a  $k$ -form and a scalar is again a  $k$ -form. The vector space of all  $k$ -forms is denoted by  $\wedge^k V^*$ :

$$\boxed{\wedge^k V^* := \{ k\text{-forms on } V^k \} \subseteq (V^*)^{\otimes k} .}$$

It is convenient to define

$$\wedge^0 V^* := \mathbb{K}.$$

Note that  $\wedge^1 V^* = (V^*)^{\otimes 1} = V^*$  since there is nothing to switch when  $k = 1$ .

**Example 2.32.** We’ve already seen many examples of  $k$ -forms, all of which having to do with determinants. As a quick reminder, recall that given any one-forms  $\alpha \in T_p^* \mathbb{R}^3$  and  $\beta \in T_p^* \mathbb{R}^3$ , we define  $\alpha \wedge \beta \in \wedge^2 T_p^* \mathbb{R}^3$  as the two-form

$$(\alpha \wedge \beta)(v_1, v_2) := \alpha(v_1)\beta(v_2) - \beta(v_1)\alpha(v_2) = \det \begin{bmatrix} \alpha(v_1) & \alpha(v_2) \\ \beta(v_1) & \beta(v_2) \end{bmatrix}$$

for all  $v_1, v_2 \in T_p \mathbb{R}^3$ . Given another one-form  $\gamma \in T_p^* \mathbb{R}^3$ , the three-form  $\alpha \wedge \beta \wedge \gamma \in \wedge^3 T_p^* \mathbb{R}^3$  is defined by

$$(\alpha \wedge \beta \wedge \gamma)(v_1, v_2, v_3) := \det \begin{bmatrix} \alpha(v_1) & \alpha(v_2) & \alpha(v_3) \\ \beta(v_1) & \beta(v_2) & \beta(v_3) \\ \gamma(v_1) & \gamma(v_2) & \gamma(v_3) \end{bmatrix}$$

for all  $v_1, v_2, v_3 \in T_p \mathbb{R}^3$ . These two examples are important as they serve as models for the general “wedge product” to be defined in Section 2.6.2.

We now discuss a slightly more complicated definition of alternating tensors using permutations. In the appendix to this section we discuss permutations, so we suggest you quickly review that appendix in case you quickly meet a foreign word. Let  $\sigma \in S_k$ , the group of permutations on  $\{1, \dots, k\}$ . Then  $\sigma$  induces a map on  $V^k = V \times \dots \times V$  ( $k$  number of  $V$ 's) as follows:

$$(2.60) \quad \sigma(v_1, v_2, \dots, v_k) := (v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}).$$

In other words,  $\sigma$  just permutes the order of  $v_1, \dots, v_k$ . We now introduce a useful operation called the **pullback** of  $\sigma$  as a map on the tensor product  $(V^*)^{\otimes k} = V^* \otimes \dots \otimes V^*$  ( $k$  number of  $V^*$ 's) defined as follows. Let  $\alpha \in (V^*)^{\otimes k}$ , which recall just means that  $\alpha : V^k \rightarrow \mathbb{K}$  is multi-linear. Then  $\sigma^* \alpha \in (V^*)^{\otimes k}$  is by definition the map  $\sigma^* \alpha : V^k \rightarrow \mathbb{K}$  defined by

$$\boxed{(\sigma^* \alpha)(v) := \alpha(\sigma v) \quad \text{for all } v \in V^k.}$$

From this formula it's not difficult to check that  $\sigma^* \alpha : V^k \rightarrow \mathbb{K}$  is multi-linear and that

$$\sigma^* : (V^*)^{\otimes k} \rightarrow (V^*)^{\otimes k}$$

is a linear map.

**Example 2.33.** Let  $i < j$  and let  $\tau \in S_k$  be the transposition  $\tau = (i \ j)$  switching  $i$  and  $j$ . Explicitly,  $\tau(i) = j$ ,  $\tau(j) = i$ , and  $\tau(k) = k$  for  $k \neq i, j$ . Then

$$\tau(v_1, \dots, v_i, \dots, v_j, \dots, v_k) := (v_1, \dots, v_j, \dots, v_i, \dots, v_k),$$

so for any  $\alpha \in (V^*)^{\otimes k}$ , we have

$$(\tau^* \alpha)(v_1, \dots, v_i, \dots, v_j, \dots, v_k) := \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

From this example, we see that  $\alpha \in (V^*)^{\otimes k}$  is alternating if and only if

$$\boxed{\tau^* \alpha = -\alpha \quad \text{for all transpositions } \tau \in S_k,}$$

because this just means for all  $(v_1, \dots, v_k) \in V^k$  and  $i < j$ , we have

$$\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) = -\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k).$$

Here are some properties of the pullback operation.

**PROPOSITION 2.26.** *The pullback reverses composition: For permutations  $\sigma, \rho$ , we have*

$$(\sigma \circ \rho)^* = \rho^* \circ \sigma^*.$$

**PROOF.** For any  $\alpha \in (V^*)^{\otimes k}$  and  $v \in V^k$ , by definition of pullback, we have

$$(\sigma \circ \rho)^* \alpha(v) = \alpha((\sigma \circ \rho)(v)) = \alpha(\sigma(\rho(v))) = \sigma^* \alpha(\rho(v)) = \rho^*(\sigma^* \alpha)(v).$$

Thus,  $(\sigma \circ \rho)^*(\alpha) = \rho^*(\sigma^*(\alpha))$ . □

Recall that transpositions are the building blocks of all permutations in the sense that any permutation  $\sigma$  can be written as a product of transpositions (see Appendix):  $\sigma = \tau_1 \tau_2 \dots \tau_\ell$ . The sign of a permutation is the number  $\text{sgn } \sigma := (-1)^\ell$ .

**THEOREM 2.27.** *A tensor  $\alpha \in (V^*)^{\otimes k}$  is alternating if and only if*

$$\boxed{\sigma^* \alpha = \text{sgn } \sigma \alpha \quad \text{for all permutations } \sigma \in S_k.}$$

*Explicitly, this means that for all  $\sigma \in S_k$  and  $(v_1, \dots, v_k) \in V^k$ , we have*

$$\alpha(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}) = \text{sgn } \sigma \alpha(v_1, v_2, \dots, v_k).$$

PROOF. To see this, observe that if  $\sigma$  is alternating, then certainly  $\tau^*\alpha = -\alpha$  for all transpositions because the sign of a transposition is  $-1$ . On the other hand, assuming this holds for all transpositions, given any permutation  $\sigma$  and writing  $\sigma = \tau_1\tau_2\cdots\tau_\ell$  as a product of transpositions, using that pullback reverses composition, we have

$$\begin{aligned}\sigma^*\alpha &= (\tau_1\tau_2\tau_3\cdots\tau_\ell)^*\alpha = \tau_\ell^*\tau_{\ell-1}^*\cdots\tau_3\tau_2^*\tau_1^*\alpha \\ &= -1 \cdot \tau_\ell^*\tau_{\ell-1}^*\cdots\tau_3\tau_2^*\alpha \\ &= (-1)^2 \cdot \tau_\ell^*\tau_{\ell-1}^*\cdots\tau_3^*\alpha = \cdots = (-1)^\ell \alpha = \text{sgn } \sigma \alpha.\end{aligned}$$

□

Our next order of business is to show how to turn  $k$ -tensors into  $k$ -forms.

**2.6.2. The notorious wedge product.** Given any  $\alpha \in (V^*)^{\otimes k}$ , we define  $\alpha_a \in (V^*)^{\otimes k}$  by the formula

$$(2.61) \quad \alpha_a := \sum_{\sigma} \text{sgn } \sigma (\sigma^*\alpha).$$

where the sum is over all  $\sigma \in S_k$ . In particular, this sum is over  $k!$  different permutations. In the following proposition we show that  $\alpha_a$  is alternating, that is,  $\alpha_a \in \wedge^k V^*$ . Therefore, the operator  $\alpha \mapsto \alpha_a$  is sometimes called the **alternating operator**.

PROPOSITION 2.28. *The alternating operator defines a surjective linear map*

$$(V^*)^{\otimes k} \ni \alpha \mapsto \alpha_a \in \wedge^k V^*.$$

*In fact, if  $\alpha \in \wedge^k V^*$ , then  $\alpha_a = k! \alpha$ .*

PROOF. To prove that  $\alpha_a$  is alternating, we just use the definition: If  $\rho \in S_k$ , then

$$\begin{aligned}\rho^*\alpha_a &:= \sum_{\sigma} \text{sgn } \sigma \rho^*(\sigma^*\alpha) = \sum_{\sigma} \text{sgn } \sigma (\sigma\rho)^*\alpha \\ &= \text{sgn } \rho \sum_{\sigma} \text{sgn}(\sigma\rho) (\sigma\rho)^*\alpha,\end{aligned}$$

where we used that  $\text{sgn}(\sigma\rho) = \text{sgn } \sigma \text{sgn } \rho$  and that  $(\text{sgn } \rho)^2 = 1$ . As the sum runs over all  $\sigma \in S_k$ , the permutation  $\sigma\rho$  runs over all permutations of  $S_k$ . Therefore, writing  $\kappa$  for  $\sigma\rho$  we can write the last sum as

$$\rho^*\alpha_a = \text{sgn } \rho \sum_{\kappa} \text{sgn } \kappa \kappa^*\alpha,$$

where the sum is over all  $\kappa \in S_k$ . However, this just means  $\rho^*\alpha_a = \text{sgn } \rho \alpha_a$  and so  $\alpha_a$  is alternating.

If  $\alpha \in \wedge^k V^*$ , then

$$\alpha_a := \sum_{\sigma} \text{sgn } \sigma (\sigma^*\alpha) = \sum_{\sigma} \text{sgn } \sigma \text{sgn } \sigma \alpha = \sum_{\sigma} \alpha = k! \alpha.$$

□

We now define the wedge product of two tensors. For  $\alpha \in (V^*)^{\otimes j}$ ,  $\beta \in (V^*)^{\otimes k}$ , we define the **wedge product** of  $\alpha$  and  $\beta$  as the alternating  $(j+k)$ -form

$$\alpha \wedge \beta := \frac{1}{j!k!} (\alpha \otimes \beta)_a.$$

Thus,

$$(V^*)^{\otimes j} \times (V^*)^{\otimes k} \ni (\alpha, \beta) \mapsto \alpha \wedge \beta \in \wedge^{j+k} V^*.$$

Using the definition of the alternating operator, we get the more explicit formula

$$(2.62) \quad \alpha \wedge \beta := \frac{1}{j!k!} \sum_{\sigma} \operatorname{sgn} \sigma \sigma^*(\alpha \otimes \beta),$$

where the sum is over all permutations in  $S_{j+k}$ . Even more explicitly (but more monstrous!),  $\alpha \wedge \beta : V^{j+k} \rightarrow \mathbb{K}$  is the map

$$(2.63) \quad \begin{aligned} & (\alpha \wedge \beta)(v_1, \dots, v_{j+k}) \\ &= \frac{1}{j!k!} \sum_{\sigma} \operatorname{sgn} \sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(j)}) \beta(v_{\sigma(j+1)}, \dots, v_{\sigma(j+k)}). \end{aligned}$$

This last expression for  $\alpha \wedge \beta$  is very intimidating and it's *not* recommended to work with when  $j$  and  $k$  are large. For small  $j, k$  (e.g.  $j, k \leq 2$ ) it's OK.

**Example 2.34.** For example, let  $\alpha, \beta \in \wedge^1 V^* = V^*$ . Then  $\alpha \wedge \beta \in \wedge^2 V$  is the alternating map

$$(\alpha \wedge \beta)(v_1, v_2) = \frac{1}{1 \cdot 1} \sum_{\sigma} \operatorname{sgn} \sigma \alpha(v_{\sigma(1)}) \beta(v_{\sigma(2)})$$

where the sum is over all permutations  $\sigma$  of  $\{1, 2\}$ . In this case there are only two such permutations, the identity  $i$  and the transposition of 1 and 2. Hence,

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1) \beta(v_2) - \alpha(v_2) \beta(v_1).$$

More succinctly,

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha.$$

You can also get this formula directly from (2.62):

$$\alpha \wedge \beta = \operatorname{sgn} i i^*(\alpha \otimes \beta) + \operatorname{sgn}(1 \ 2) (1 \ 2)^*(\alpha \otimes \beta) = \alpha \otimes \beta - \beta \otimes \alpha.$$

The following theorem, which lists some important properties of  $\wedge$ , is sometimes left as a project for the reader and other times is stated as “immediate from the definition (2.63)”; the first is being mean and the second is not honest (unless “immediate” means after many hours of work figuring out the monster (2.63))!

**THEOREM 2.29.** *The wedge product has the following properties: For  $\alpha \in (V^*)^{\otimes j}$ ,  $\beta \in (V^*)^{\otimes k}$ , and  $\gamma \in (V^*)^{\otimes \ell}$ , we have*

- (i)  $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ .
- (ii)  $(a\alpha + \beta) \wedge \gamma = a\alpha \wedge \gamma + \beta \wedge \gamma$  for  $j = k$  and  $a \in \mathbb{K}$ .
- (iii)  $\alpha \wedge \beta = (-1)^{jk} \beta \wedge \alpha$ .

*In other words, wedging is associative, distributive, and “anti-” commutative (also called “super” commutative).*

PROOF. (ii) is (honestly) easy so we'll leave that one to you.

*Proof of (i):* We shall prove that

$$(2.64) \quad (\alpha \wedge \beta) \wedge \gamma = \frac{1}{j! k! \ell!} \sum_{\kappa} \operatorname{sgn} \kappa \kappa^*(\alpha \otimes \beta \otimes \gamma) = \alpha \wedge (\beta \wedge \gamma),$$

where the sum is over all permutations in  $S_{j+k+\ell}$ . Consider the first equality; the second equality is proved in a similar way. Using the definition of wedge product on  $(\alpha \wedge \beta) \wedge \gamma$ , we obtain

$$\begin{aligned} (\alpha \wedge \beta) \wedge \gamma &= \frac{1}{(j+k)! \ell!} \sum_{\sigma} \operatorname{sgn} \sigma \sigma^*((\alpha \wedge \beta) \otimes \gamma) \\ &= \frac{1}{(j+k)! j! k! \ell!} \sum_{\sigma} \sum_{\rho} \operatorname{sgn} \sigma \operatorname{sgn} \rho \sigma^*(\rho^*(\alpha \otimes \beta) \otimes \gamma), \end{aligned}$$

where the outer sum is over  $\sigma \in S_{j+k+\ell}$  and the inner one over  $\rho \in S_{j+k}$ . For  $\rho \in S_{j+k}$ , define  $\tilde{\rho} \in S_{j+k+\ell}$  as  $\tilde{\rho} = \rho$  on  $\{1, 2, \dots, j+k\}$  and  $\tilde{\rho}$  is the identity on  $\{j+k+1, \dots, j+k+\ell\}$ . Because transpositions defining  $\rho$  also define  $\tilde{\rho}$ , we have  $\operatorname{sgn} \tilde{\rho} = \operatorname{sgn} \rho$ . Also, because  $\tilde{\rho}$  is the identity on  $\{j+k+1, \dots, j+k+\ell\}$ , we have

$$\rho^*(\alpha \otimes \beta) \otimes \gamma = \tilde{\rho}^*(\alpha \otimes \beta \otimes \gamma).$$

Thus,

$$\begin{aligned} (\alpha \wedge \beta) \wedge \gamma &= \frac{1}{(j+k)! j! k! \ell!} \sum_{\sigma} \sum_{\rho} \operatorname{sgn} \sigma \operatorname{sgn} \tilde{\rho} \sigma^* \tilde{\rho}^*(\alpha \otimes \beta \otimes \gamma) \\ &= \frac{1}{(j+k)! j! k! \ell!} \sum_{\sigma} \sum_{\rho} \operatorname{sgn}(\tilde{\rho}\sigma) (\tilde{\rho}\sigma)^*(\alpha \otimes \beta \otimes \gamma) \\ &= \frac{1}{(j+k)! j! k! \ell!} \sum_{\rho} \sum_{\sigma} \operatorname{sgn}(\tilde{\rho}\sigma) (\tilde{\rho}\sigma)^*(\alpha \otimes \beta \otimes \gamma). \end{aligned}$$

Now for fixed  $\rho$ , the inner sum runs over all  $\sigma \in S_{j+k+\ell}$  and hence the permutation  $\tilde{\rho}\sigma$  also runs over all permutations of  $S_{j+k+\ell}$ . Therefore, writing  $\kappa$  for  $\tilde{\rho}\sigma$  we can write the last sum as

$$(\alpha \wedge \beta) \wedge \gamma = \frac{1}{(j+k)! j! k! \ell!} \sum_{\rho} \sum_{\kappa} \operatorname{sgn} \kappa \kappa^*(\alpha \otimes \beta \otimes \gamma),$$

where the inner sum is over all  $\kappa \in S_{j+k+\ell}$ . Now there is no dependence on  $\rho$ . Therefore, since there are exactly  $(j+k)!$  elements of  $S_{j+k}$ , we have

$$(\alpha \wedge \beta) \wedge \gamma = \frac{1}{j! k! \ell!} \sum_{\kappa} \operatorname{sgn} \kappa \kappa^*(\alpha \otimes \beta \otimes \gamma),$$

which is exactly (2.64). This proves (i).

*Proof of (iii):* By definition of  $\wedge$ , we have

$$\alpha \wedge \beta = \frac{1}{j! k!} \sum_{\sigma} \operatorname{sgn} \sigma \sigma^*(\alpha \otimes \beta)$$

and

$$\beta \wedge \alpha = \frac{1}{j! k!} \sum_{\sigma} \operatorname{sgn} \sigma \sigma^*(\beta \otimes \alpha).$$

Define  $\rho \in S_{j+k}$  by the table (2.76) found in the appendix to this section:

$$\begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & j+k-1 & j+k \\ \hline 1+k & 2+k & \cdots & j-1+k & j+k & 1 & \cdots & k-1 & k \end{array}$$

Then observe that

$$\alpha \otimes \beta = \rho^*(\beta \otimes \alpha).$$

Now using that  $\text{sgn } \rho \cdot \text{sgn } \rho = 1$  and  $\text{sgn}(\sigma\rho) = \text{sgn } \sigma \text{sgn } \rho$ , observe that

$$\begin{aligned} \alpha \wedge \beta &= \frac{1}{j!k!} \sum_{\sigma} \text{sgn } \sigma \sigma^* \rho^*(\beta \otimes \alpha) \\ &= \frac{1}{j!k!} \text{sgn } \rho \sum_{\sigma} \text{sgn}(\rho\sigma) (\rho\sigma)^*(\beta \otimes \alpha) \\ &= \frac{1}{j!k!} \text{sgn } \rho \sum_{\kappa} \text{sgn } \kappa \kappa^*(\beta \otimes \alpha), \end{aligned}$$

where we put  $\kappa = \sigma\rho$ . Therefore,

$$\alpha \wedge \beta = \text{sgn } \rho \beta \wedge \alpha = (-1)^{jk} \beta \wedge \alpha,$$

since in Example 2.35 of the appendix to this section we proved that  $\text{sgn } \rho = (-1)^{jk}$ .  $\square$

As a consequence of (iii), we get

**COROLLARY 2.30.** *For any odd degree tensor  $\alpha$ , we have  $\alpha \wedge \alpha = 0$ , and the wedge product is commutative when one of the tensors is of even degree.*

**PROOF.** With  $\beta = \alpha$ , where  $\alpha$  is of odd degree, we have

$$\alpha \wedge \alpha = \alpha \wedge \beta = (-1)^{\text{odd number}} \beta \wedge \alpha = -\alpha \wedge \alpha.$$

Therefore,  $\alpha \wedge \alpha = 0$ . That the wedge product is commutative when one of the tensors is of even degree follows from the fact that  $(-1)^{jk} = +1$  if  $j$  or  $k$  is even.  $\square$

By associativity, the wedge of any number of forms is well-defined without the use of parentheses; thus,

$$\alpha \wedge \beta \wedge \gamma \text{ is unambiguously defined as } (\alpha \wedge \beta) \wedge \gamma \text{ or } \alpha \wedge (\beta \wedge \gamma),$$

the same can be said of any number of wedges. Moreover, as we saw in Equation (2.64) of the above proof, if  $\alpha, \beta$ , and  $\gamma$  are  $j, k$ , and  $\ell$  forms, then

$$\alpha \wedge \beta \wedge \gamma = \frac{1}{j!k!\ell!} \sum_{\sigma} \text{sgn } \sigma \sigma^*(\alpha \otimes \beta \otimes \gamma),$$

where the sum is over all permutations  $\kappa \in S_{j+k+\ell}$ . One can use induction to prove that if  $\alpha_1, \dots, \alpha_k$  are  $i_1, \dots, i_k$ -forms, then

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = \frac{1}{i_1! i_2! \cdots i_k!} \sum_{\sigma} \text{sgn } \sigma \sigma^*(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k),$$

where the sum is over all permutations  $\sigma \in S_{i_1+\dots+i_k}$ . In terms of the alternating operator, we can write this as

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = \frac{1}{i_1! i_2! \cdots i_k!} (\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k)_a.$$

Fortunately we won't have too many occasions where we need this more general formula except when all the  $\alpha_j$ 's are one-forms, in which case it takes the simpler form:

PROPOSITION 2.31. *If  $\alpha_1, \dots, \alpha_k \in \wedge^1 V^*$ , then*

$$(2.65) \quad \boxed{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = \sum_{\sigma} \operatorname{sgn} \sigma \alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)} \otimes \cdots \otimes \alpha_{\sigma(k)},}$$

where the sum is over all  $\sigma \in S_k$ .

PROOF. For  $v = (v_1, \dots, v_k) \in V^k$ , we have

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k)(v) = \sum_{\sigma} \operatorname{sgn} \sigma \alpha_1(v_{\sigma(1)}) \cdot \alpha_2(v_{\sigma(2)}) \cdots \alpha_k(v_{\sigma(k)}).$$

We claim that

$$\alpha_1(v_{\sigma(1)}) \cdot \alpha_2(v_{\sigma(2)}) \cdots \alpha_k(v_{\sigma(k)}) = \alpha_{\sigma^{-1}(1)}(v_1) \cdot \alpha_{\sigma^{-1}(2)}(v_2) \cdots \alpha_{\sigma^{-1}(k)}(v_k)$$

To see this, let  $1 \leq j \leq k$  and consider the  $v_j$  term on the left-hand side of this equation. Let  $1 \leq i \leq n$  be such that  $\sigma(i) = j$ , that is,  $i = \sigma^{-1}(j)$ . Then,

$$\alpha_i(v_{\sigma(i)}) = \alpha_{\sigma^{-1}(j)}(v_j),$$

which is exactly the  $v_j$  term on the right-hand side of the above equation. Therefore,

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k)(v) = \sum_{\sigma} \operatorname{sgn} \sigma (\alpha_{\sigma^{-1}(1)} \otimes \alpha_{\sigma^{-1}(2)} \otimes \cdots \otimes \alpha_{\sigma^{-1}(k)})(v).$$

Putting  $\rho = \sigma^{-1}$  and using that  $\operatorname{sgn} \sigma = \operatorname{sgn} \rho$ , we get our result.  $\square$

**2.6.3. The exterior algebra.** We now focus just on forms. Thus, interpreting Theorem 2.29 for forms only (although it holds for any tensors) we have: The wedge product defines a map

$$\wedge^j V^* \times \wedge^k V^* \ni (\alpha, \beta) \mapsto \alpha \wedge \beta \in \wedge^{j+k} V^*$$

and has the following properties for  $\alpha \in \wedge^j V^*$ ,  $\beta \in \wedge^k V^*$ , and  $\gamma \in \wedge^\ell V^*$ :

- (i)  $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ .
- (ii)  $(a\alpha + \beta) \wedge \gamma = a\alpha \wedge \gamma + \beta \wedge \gamma$  for  $j = k$  and  $a \in \mathbb{K}$ .
- (iii)  $\alpha \wedge \beta = (-1)^{jk} \beta \wedge \alpha$ .

When  $j$  or  $k$  is zero (so  $\wedge^j V^* = \mathbb{K}$  or  $\wedge^k V^* = \mathbb{K}$ ), we interpret  $\wedge$  as scalar multiplication. If  $V$  is an  $n$ -dimensional vector space, then in Theorem 2.33 below we prove that  $\wedge^k V^* = 0$  for  $k > n$ . The remaining spaces for  $k \leq n$  form the **exterior algebra**, which is by definition the space

$$\wedge V^* := \wedge^0 V \oplus \wedge^1 V^* \oplus \wedge^2 V^* \oplus \cdots \oplus \wedge^n V^* \quad \text{where } \wedge^0 V = \mathbb{K};$$

this space is a ‘‘super’’ algebra over  $\mathbb{K}$  with the product given by the wedge product. The word ‘‘super’’ is used because of the factor  $(-1)^{jk}$  appearing in (iii) whenever we try to commute two forms.

We now introduce some useful notation. Let  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  be a basis for  $V^*$  and let  $I = (i_1, \dots, i_k)$  be a  $k$ -tuple of integers where  $i_1, \dots, i_k \in \{1, \dots, n\}$ . We define

$$\alpha_I := \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_k}.$$

LEMMA 2.32. *Let  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  be a basis for  $V^*$  and let  $I = (i_1, \dots, i_k)$  where  $i_1, \dots, i_k \in \{1, \dots, n\}$ . Then,*

- (i) If  $k > n$  or if  $i_1, \dots, i_k$  are not all distinct (that is, if there is an integer in this list appearing more than once), then  $\alpha_I = 0$ .
- (ii) Let  $1 \leq k \leq n$ , assume that  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ , and let  $\{v_1, v_2, \dots, v_n\}$  be the dual basis for  $V^{**} = V$  (so that  $\alpha_i(v_j) = \delta_{ij}$  for all  $i, j$ ). Then,

$$\alpha_I(v_J) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J, \end{cases}$$

where  $J = (j_1, \dots, j_k)$  with  $1 \leq j_1 < \dots < j_k \leq n$  and  $v_J = (v_{j_1}, \dots, v_{j_k})$ .

PROOF. To prove the first claim observe that if  $k > n$ , then there must be a repeated index in  $\alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k}$ . Thus, we just have to prove the statement concerning the repeated index; let us say we have  $i_r = i_s$  with  $r < s$ :

$$(2.66) \quad \alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k} = \alpha \wedge \alpha_{i_r} \wedge \beta \wedge \alpha_{i_s} \wedge \gamma,$$

where  $\alpha, \beta, \gamma$  are wedges of some of the  $\alpha_j$ 's. By anti-commutativity, we have  $\beta \wedge \alpha_{i_s} = \pm \alpha_{i_s} \wedge \beta$  and also  $\alpha_{i_r} \wedge \alpha_{i_s} = 0$ . Therefore,

$$\alpha \wedge \alpha_{i_r} \wedge \beta \wedge \alpha_{i_s} \wedge \gamma = \pm \alpha \wedge \alpha_{i_r} \wedge \alpha_{i_s} \wedge \beta \wedge \gamma = \pm \alpha \wedge 0 \wedge \beta \wedge \gamma = 0,$$

hence,  $\alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k} = 0$  if some  $i_r = i_s$ .

To prove the second claim, we use the formula (2.65):

$$\begin{aligned} (\alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k})(v_{j_1}, v_{j_2}, \dots, v_{j_k}) &= \sum_{\sigma} \text{sgn } \sigma \alpha_{i_{\sigma(1)}}(v_{j_1}) \alpha_{i_{\sigma(2)}}(v_{j_2}) \dots \alpha_{i_{\sigma(k)}}(v_{j_k}) \\ &= \sum_{\sigma} \text{sgn } \sigma \delta_{i_{\sigma(1)}, j_1} \delta_{i_{\sigma(2)}, j_2} \dots \delta_{i_{\sigma(k)}, j_k} \end{aligned}$$

If  $(i_1, \dots, i_k) = (j_1, \dots, j_k)$ , then there is exactly one term in this sum not zero, the term with  $\sigma$  the identity, therefore we get 1 for the sum in this case. On the other hand, if  $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$ , say  $j_r$  is not amongst the  $i_1, \dots, i_k$ , then  $i_{\sigma(r)} \neq j_r$  no matter what  $\sigma$  is, hence we get 0 for the sum in this case.  $\square$

**THEOREM 2.33.** *If  $V$  is an  $n$ -dimensional vector space, then  $\wedge^k V^* = \{0\}$  for  $k > n$  and  $\wedge V^*$  is finite-dimensional with  $\dim \wedge V^* = 2^n$ . In fact,*

$$\dim(\wedge^k V^*) = \binom{n}{k}, \quad 0 \leq k \leq n,$$

and if  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is a basis for  $V^*$ , then the set of all wedge products of the form

$$(2.67) \quad \{\alpha_I \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$

is a basis for  $\wedge^k V^*$  (for  $1 \leq k \leq n$ ).

PROOF. We first prove that any given  $\alpha \in \wedge^k V^*$  can be written in terms of the  $\alpha_I$ 's. To do so, note that since  $\alpha \in \wedge^k V^* \subseteq (V^*)^{\otimes k}$  and we already know that  $\{\alpha_{i_1} \otimes \alpha_{i_2} \otimes \dots \otimes \alpha_{i_k}\}$ , where  $i_1, \dots, i_k$  run over all elements of  $\{1, \dots, n\}$ , is a basis for  $(V^*)^{\otimes k}$ , we can write

$$\alpha = \sum a_I \alpha_{i_1} \otimes \alpha_{i_2} \otimes \dots \otimes \alpha_{i_k},$$

where the sum is over all  $I = (i_1, \dots, i_k)$  with  $i_1, \dots, i_k \in \{1, \dots, n\}$ . Applying the alternating map, using that  $\alpha_a = k! \alpha$  (by Proposition 2.28), and dividing by  $k!$ , we obtain

$$\alpha = \frac{1}{k!} \alpha_a = \sum a_I \frac{1}{k!} (\alpha_{i_1} \otimes \alpha_{i_2} \otimes \dots \otimes \alpha_{i_k})_a = \sum \frac{a_I}{k!} \alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k}.$$

Thus,

$$\alpha = \sum a_I \alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k}$$

where the sum is over all  $i_1, \dots, i_k \in \{1, \dots, n\}$ . By our lemma we know that if  $k > n$ , then the right-hand side is zero, therefore  $\wedge^k V^* = \{0\}$  for all  $k > n$ . Thus, for the remainder of the proof we assume that  $1 \leq k \leq n$ . Fix an  $i_1, \dots, i_k$  with  $i_1, \dots, i_k \in \{1, \dots, n\}$ , then order and relabel them such that  $j_1 < j_2 < \dots < j_k$ . We claim that

$$\alpha_{i_1} \wedge \alpha_{i_2} \wedge \dots \wedge \alpha_{i_k} = \pm \alpha_{j_1} \wedge \alpha_{j_2} \wedge \dots \wedge \alpha_{j_k}$$

This is ‘‘obvious’’ because we can transpose the  $i_r$ ’s to reorder them to get the  $j_r$ ’s. Here are the details for the pedantic ones. For each  $r$  we know that  $j_r = i_s$  for a unique  $s$ ; let  $\rho$  take  $r \mapsto s$ ; thus  $\rho$  is a permutation on  $\{1, \dots, k\}$  such that  $j_r = i_{\rho(r)}$  for  $r = 1, \dots, k$ . Then one can check that  $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} = \rho^*(\alpha_{j_1} \otimes \dots \otimes \alpha_{j_k})$ , so

$$\begin{aligned} \alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} &= \sum_{\sigma} \operatorname{sgn} \sigma \sigma^*(\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k}) \\ &= \sum_{\sigma} \operatorname{sgn} \sigma \sigma^*(\rho^*(\alpha_{j_1} \otimes \dots \otimes \alpha_{j_k})) \\ &= \operatorname{sgn} \rho \sum_{\sigma} \operatorname{sgn}(\rho\sigma) (\rho\sigma)^*(\alpha_{j_1} \otimes \dots \otimes \alpha_{j_k}) \\ &= \operatorname{sgn} \rho \sum_{\kappa} \operatorname{sgn} \kappa \kappa^*(\alpha_{j_1} \otimes \dots \otimes \alpha_{j_k}) = \operatorname{sgn} \rho \alpha_{j_1} \wedge \dots \wedge \alpha_{j_k}. \end{aligned}$$

This proves  $\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} = \pm \alpha_{j_1} \wedge \dots \wedge \alpha_{j_k}$ , so we can write

$$\alpha = \sum_{i_1 < \dots < i_k} b_I \alpha_I$$

for some constants  $b_I$ . In particular, the set (2.67) certainly spans  $\wedge^k V^*$ . We now prove that the set (2.67) is linearly independent. Thus, assume that

$$\sum_{i_1 < \dots < i_k} a_I \alpha_I = 0.$$

Let  $\{v_j\}$  denote the dual basis to  $\{\alpha_i\}$ . Then taking any  $1 \leq j_1 \leq \dots \leq j_k \leq n$ , putting  $v_J := (v_{j_1}, \dots, v_{j_k})$ , and using Lemma 2.32 we see that

$$0 = 0(v_J) = \sum_{i_1 < \dots < i_k} a_I \alpha_I(v_J) = a_J.$$

Since  $1 \leq j_1 \leq \dots \leq j_k \leq n$  were arbitrary, this shows that all the  $a_I$ ’s are zero. Therefore, the set (2.67) is indeed a basis. Since  $\binom{n}{k}$  is the number of subsets of  $k$  elements from an  $n$  element set, it follows that the set (2.67) has exactly  $\binom{n}{k}$  elements. Hence,  $\dim \wedge^k V^* = \binom{n}{k}$ ; this formula even holds for  $k = 0$  since  $\binom{n}{0} = 1$ . Finally, recall the binomial formula:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Applying this formula with  $x = y = 1$ , we see that

$$\dim \wedge V = \sum_{k=0}^n \dim \wedge^k V = \sum_{k=0}^n \binom{n}{k} = (1+1)^n = 2^n.$$

□

Note, in particular, that  $\wedge^n V^*$  is one-dimensional and a basis is  $\alpha_1 \wedge \cdots \wedge \alpha_n$  for any basis  $\{\alpha_i\}$  of  $V^*$ .

**2.6.4. The determinant.** A neat application of the exterior algebra is to give an invariant definition of the determinant of a linear map.

Let  $V$  be an  $n$ -dimensional  $\mathbb{K}$ -vector space and let  $f : V \rightarrow V$  be linear. Then for any  $k$ , the map  $f$  induces a map  $f : V^k \rightarrow V^k$  via

$$f(v_1, \dots, v_k) := (fv_1, \dots, fv_k) \quad \text{for all } (v_1, \dots, v_k) \in V^k.$$

The **pullback**  $f^*$  of this map can be defined just as we did in Section 2.6.1 for permutations:

$$f^* : \wedge^k V^* \rightarrow \wedge^k V^*$$

is defined as follows: If  $\alpha \in \wedge^k V^*$ , then  $f^*\alpha \in \wedge^k V^*$  is defined by

$$(f^*\alpha)(v) := \alpha(fv) \quad \text{for all } v \in V^k.$$

We are really interested in  $k = n$ . In this case we know that  $\wedge^n V^*$  is one-dimensional, and therefore

$$f^* : \wedge^n V^* \rightarrow \wedge^n V^*$$

is just multiplication by a number  $a \in \mathbb{K}$  (see, for example, Problem 1 in Exercises 2.5). In other words,  $f^*\alpha = a\alpha$  for all  $\alpha \in \wedge^n V^*$ . The number  $a$  is by definition the **determinant** of  $f$  and is denoted by  $\det f$ ; thus,

$$(2.68) \quad \boxed{f^*\alpha = \det f \alpha \quad \text{for all } \alpha \in \wedge^n V^*.$$

We can relate this expression to the well-known expression for the determinant as seen in linear algebra classes. Let  $\{w_j\}_{j=1}^n$  be a basis for  $V$ , let  $\{\alpha_i\}$  be the dual basis to  $\{w_i\}$  and consider  $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_n$  and  $v = (w_1, \dots, w_n)$ . Then by Lemma 2.32, we have  $\alpha(v) = 1$ , so

$$(2.69) \quad \det f = \det f \alpha(v) = f^*\alpha(v) = \alpha(fv) = \alpha(fw_1, \dots, fw_n).$$

Now write

$$(2.70) \quad f(w_j) = \sum_{i=1}^m a_{ij} w_i, \quad j = 1, \dots, n,$$

for unique constants  $a_{ij}$ . Then by the formula (see (2.65))

$$\alpha = \sum_{\sigma} \operatorname{sgn} \sigma \alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)} \otimes \cdots \otimes \alpha_{\sigma(n)},$$

we have

$$\alpha(fw_1, \dots, fw_n) = \sum_{\sigma} \operatorname{sgn} \sigma \alpha_{\sigma(1)}(fw_1) \cdot \alpha_{\sigma(2)}(fw_2) \cdots \alpha_{\sigma(n)}(fw_n).$$

Finally, by (2.70) we see that  $\alpha_{\sigma(j)}(fw_j) = a_{\sigma(j)j}$ , and so

$$(2.71) \quad \boxed{\det f = \sum_{\sigma} \operatorname{sgn} \sigma a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n},$$

the well-known definition of determinant for a matrix  $[a_{ij}]$ ! What is so beautiful about the formula (2.68) is that it's basis independent. This shows that the determinant is a God-given (or intrinsic) property associated to linear maps, which is far from obvious in the traditional definition (2.71). The definition (2.68) can also be used to easily derive many properties of the determinant that are messy with the definition (2.71). Consider for example the proofs in the following theorem.

**THEOREM 2.34.** *The determinant function  $\det : \text{hom}(V) \rightarrow \mathbb{K}$  has the following properties for  $f, g \in \text{hom}(V)$ :*

- (i)  $\det \text{Id} = 1$ .
- (ii)  $\det(fg) = \det f \cdot \det g$  (the determinant is multiplicative).
- (iii)  $f$  is bijective if and only if  $\det f \neq 0$ , in which case  $\det(f^{-1}) = (\det f)^{-1}$ .
- (iv)  $\det(\sigma f) = \det(f\sigma) = \text{sgn } \sigma \cdot \det f$  for any permutation  $\sigma$  of a basis of  $V$ .

**PROOF.** We'll explain (iv) once we get to its proof.

By definition of pullback,  $\text{Id}^* \alpha = \alpha$  for all  $\alpha \in \wedge^n V^*$ . Hence,  $\det \text{Id} = 1$ .

For  $f, g \in \text{hom}(V)$  and  $\alpha \in \wedge^n V^*$ , using the definition of  $\det$  and that the pullback reverses composition (the proof is the same as in Proposition 2.26), we have

$$\det(fg) \alpha = (fg)^* \alpha = g^*(f^* \alpha) = (\det g) f^* \alpha = (\det g)(\det f) \alpha.$$

This proves (ii).

If  $f$  is bijective, then applying  $\det$  to both sides of  $f \circ f^{-1} = \text{Id}$  and using (i) and (ii), we get  $\det f \neq 0$  and  $\det(f^{-1}) = (\det f)^{-1}$ . To prove that if  $\det f \neq 0$ , then  $f$  is bijective, we shall prove the contrapositive: If  $f$  is not bijective, then  $\det f = 0$ . In this case, by the dimension theorem there is a nonzero  $v \in V$  such that  $fv = 0$ . Choose a basis  $\{w_i\}$  of  $V$  such that  $w_1 = v$ . Then  $fw_1 = 0$  so by (2.69), we have  $\det f = 0$ .

We now come to (iv). Fix any basis  $\{w_i\}$  of  $V$ . Given a permutation  $\sigma \in S_n$ , it induces a linear map on  $V$  by defining  $\sigma(w_j) := w_{\sigma(j)}$  for  $j = 1, \dots, n$  and extending by linearity; that is, if  $v = \sum a_j w_j$ , then

$$\sigma(v) := \sum a_j w_{\sigma(j)}.$$

By (ii), to prove (iv) we just have to prove that  $\det \sigma = \text{sgn } \sigma$ . To prove this is easy: Using the formula (2.69) (see discussion above (2.69) for the definition of  $\alpha$ ), we see that

$$\det \sigma = (\sigma^* \alpha)(w_1, \dots, w_n) = \text{sgn } \sigma \alpha(w_1, \dots, w_n) = \text{sgn } \sigma,$$

where we used that  $\sigma^* \alpha = \text{sgn } \sigma \alpha$  because  $\alpha$  is alternating.  $\square$

Notice that hidden in (iv) is the well-known fact that the determinant of a matrix changes sign whenever two rows or two columns of the matrix are switched.

**2.6.5. The contraction or interior product operator.** Let  $v \in V$  and fix a positive integer  $k$ . **Contraction** or the **interior product** by  $v$  is the linear map

$$\iota_v : \wedge^k V^* \rightarrow \wedge^{k-1} V^*$$

defined as follows: If  $\alpha \in \wedge^k V^*$ , then  $\iota_v \alpha \in \wedge^{k-1} V^*$  is defined by

$$(\iota_v \alpha)(w) = \alpha(v, w) \quad \text{for all } w \in V^{k-1}.$$

The operator  $\iota_v$  depends on the form degree  $k$  but this is usually omitted. The following theorem contains the main property of the contraction operator that we shall need in the sequel.

**THEOREM 2.35.** *The contraction operator  $\iota_v$  is an anti- or “super” derivation in the sense that for  $\alpha \in \wedge^j V^*$  and  $\beta \in \wedge^k V^*$ , we have*

$$(2.72) \quad \boxed{\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^j \alpha \wedge (\iota_v \beta).}$$

Thus, when  $\iota_v$  “passes over”  $\alpha$  in the second term, we have to put in a factor  $(-1)^j$  where  $j$  is the degree of  $\alpha$ .

**PROOF.** We prove this theorem in two steps.

*Step 1:* We first prove that if  $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_\ell$  with  $\gamma_i$  elements of  $V^*$ , then

$$(2.73) \quad \iota_v(\gamma_1 \wedge \cdots \wedge \gamma_\ell) = \sum_{m=1}^{\ell} (-1)^{m-1} \gamma_m(v) \gamma_1 \wedge \cdots \wedge \widehat{\gamma}_m \wedge \cdots \wedge \gamma_\ell,$$

where the hat “ $\widehat{\phantom{x}}$ ” means to omit the corresponding term; thus,

$$\gamma_1 \wedge \cdots \wedge \widehat{\gamma}_m \wedge \cdots \wedge \gamma_\ell = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \gamma_{m+1} \wedge \cdots \wedge \gamma_\ell.$$

To prove (2.73) recall from (2.65) that

$$\gamma_1 \wedge \cdots \wedge \gamma_\ell = \sum_{\sigma} \text{sgn } \sigma \gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(\ell)},$$

where the sum is over all permutations of  $\{1, 2, \dots, \ell\}$ . Hence, for any  $w \in V^{\ell-1}$ , we have

$$\begin{aligned} \iota_v(\gamma_1 \wedge \cdots \wedge \gamma_\ell)(w) &:= (\gamma_1 \wedge \cdots \wedge \gamma_\ell)(v, w) \\ &= \sum_{\sigma} \text{sgn } \sigma \gamma_{\sigma(1)}(v) \cdot (\gamma_{\sigma(2)} \otimes \cdots \otimes \gamma_{\sigma(\ell)})(w). \end{aligned}$$

We can break this up into  $\ell$  different sums:

$$\begin{aligned} \iota_v(\gamma_1 \wedge \cdots \wedge \gamma_\ell)(w) &= \sum_{\sigma(1)=1} \text{sgn } \sigma \gamma_1(v) \cdot (\gamma_{\sigma(2)} \otimes \cdots \otimes \gamma_{\sigma(\ell)})(w) \\ &+ \sum_{\sigma(1)=2} \text{sgn } \sigma \gamma_2(v) \cdot (\gamma_{\sigma(2)} \otimes \cdots \otimes \gamma_{\sigma(\ell)})(w) \\ &+ \cdots + \sum_{\sigma(1)=\ell} \text{sgn } \sigma \gamma_\ell(v) \cdot (\gamma_{\sigma(2)} \otimes \cdots \otimes \gamma_{\sigma(\ell)})(w), \end{aligned}$$

where  $\sum_{\sigma(1)=m}$  means to sum over all permutations  $\sigma$  of  $\{1, 2, \dots, \ell\}$  such that  $\sigma(1) = m$ . To prove the equality (2.73) all we have to do is show that for any  $m = 1, 2, \dots, \ell$ , we have

$$\sum_{\sigma(1)=m} \text{sgn } \sigma \gamma_{\sigma(2)} \otimes \cdots \otimes \gamma_{\sigma(\ell)} = (-1)^{m-1} \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \gamma_{m+1} \wedge \cdots \wedge \gamma_\ell.$$

To prove this, for any permutation  $\sigma$  of  $\{1, 2, \dots, \ell\}$  such that  $\sigma(1) = m$ , define a new permutation  $\rho$  by

$$\rho = \sigma(1 \ 2)(2 \ 3)(3 \ 4) \cdots (m-2 \ m-1)(m-1 \ m).$$

Observe that  $\text{sgn } \rho = \text{sgn } \sigma (-1)^{m-1}$ ,  $\rho(m) = m$ , and

$$\rho(1) = \sigma(2), \rho(2) = \sigma(3), \dots, \rho(m-1) = \sigma(m),$$

$$\rho(m+1) = \sigma(m+1), \dots, \rho(\ell-1) = \sigma(\ell-1), \rho(\ell) = \sigma(\ell).$$

Thus,

$$\begin{aligned} & \sum_{\sigma(1)=m} \operatorname{sgn} \sigma \gamma_{\sigma(2)} \otimes \gamma_{\sigma(3)} \otimes \cdots \otimes \gamma_{\sigma(\ell)} \\ &= (-1)^{m-1} \sum_{\rho(m)=m} \operatorname{sgn} \rho \gamma_{\rho(1)} \otimes \gamma_{\rho(2)} \otimes \cdots \otimes \gamma_{\rho(m-1)} \otimes \gamma_{\rho(m+1)} \otimes \cdots \otimes \gamma_{\rho(\ell)}, \end{aligned}$$

where the sum is over all permutations  $\rho$  of  $\{1, 2, \dots, \ell\}$  such that  $\rho(m) = m$ . However, the set of all such permutations is in bijective correspondence to the set of all permutations of  $\{1, 2, \dots, m-1, m+1, \dots, \ell\}$ . Hence, we can rewrite the sum as

$$= (-1)^{m-1} \sum_{\kappa} \operatorname{sgn} \kappa \gamma_{\kappa(1)} \otimes \gamma_{\kappa(2)} \otimes \cdots \otimes \gamma_{\kappa(m-1)} \otimes \gamma_{\kappa(m+1)} \otimes \cdots \otimes \gamma_{\kappa(\ell)}$$

where the sum is over all permutations  $\kappa$  of  $\{1, 2, \dots, m-1, m+1, \dots, \ell\}$ . Now this is just  $(-1)^{m-1} \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \gamma_{m+1} \wedge \cdots \wedge \gamma_{\ell}$ , and *Step 1* is proved.

*Step 2:* The formula (2.72) is multi-linear in  $\alpha$  and  $\beta$  (and also in  $v$ ) so by our basic principle (2.53) we just have to check (2.72) when  $\alpha$  and  $\beta$  are basis vectors of  $\wedge^j V^*$  and  $\wedge^k V^*$ . In fact, it suffices to check (2.72) for wedge products of the form  $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_j$  and  $\beta = \beta_1 \wedge \cdots \wedge \beta_k$  where each  $\alpha_i, \beta_i$  are elements of  $V^*$  since basis vectors of  $\wedge^j V^*$  and  $\wedge^k V^*$  are of this form. In this case, using (2.73) of *Step 1*, we have

$$\begin{aligned} \iota_v(\alpha \wedge \beta) &= \iota_v(\alpha_1 \wedge \cdots \wedge \alpha_j \wedge \beta_1 \wedge \cdots \wedge \beta_k) \\ &= \sum_{m=1}^j (-1)^{m-1} \alpha_m(v) \alpha_1 \wedge \cdots \wedge \widehat{\alpha}_m \wedge \cdots \wedge \alpha_j \wedge \beta_1 \wedge \cdots \wedge \beta_k \\ &\quad + \sum_{m=1}^k (-1)^{j+m-1} \beta_m(v) \alpha_1 \wedge \cdots \wedge \alpha_j \wedge \beta_1 \wedge \cdots \wedge \widehat{\beta}_m \wedge \cdots \wedge \beta_k \\ &= (\iota_v \alpha) \wedge \beta + (-1)^j \alpha \wedge (\iota_v \beta). \end{aligned}$$

□

**2.6.6. Last words.** Throughout this section we have been working with  $\wedge^k V^*$ , the wedge product of the dual space of a vector space  $V$ . This is only because in the next section we shall be interested in  $\wedge^k T_p^* M = \wedge^k (T_p M)^*$  so we wanted to keep notation consistent in the main text. However, one can define  $\wedge^k V$  just by replacing  $V^*$  with  $V$  in the main text. In the exercises we shall mostly use the notation  $\wedge^k V$  just so that we don't have to carry the “\*” with us everywhere.

#### EXERCISES 2.6.

1. Here are various wedge product exercises.

- (i) Prove that vectors  $v_1, \dots, v_k \in V$  are independent if and only if  $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$ . Suggestion: For necessity, complete the vectors  $\{v_1, \dots, v_k\}$  to a basis for  $V$ .
- (ii) Let  $v \in V$  be nonzero and let  $w \in \wedge^k V$ . Prove that  $v \wedge w = 0$  if and only if there is a  $u \in \wedge^{k-1} V$  such that  $w = v \wedge u$ .
- (iii) A form  $v \in \wedge^k V$  is called **decomposable** if  $v = v_1 \wedge v_2 \wedge \cdots \wedge v_k$  for some  $v_i \in V$ . Prove that if  $\dim V \leq 3$ , then every form is decomposable. In general, prove that every form in  $\wedge^{n-1} V$  and  $\wedge^n V$  is decomposable where  $n = \dim V$ .

- (iv) However, not every form is decomposable: Let  $n = 4$  and let  $v_1, v_2, v_3, v_4$  be a basis for  $V$ ; prove that  $v_1 \wedge v_2 + v_3 \wedge v_4$  is not decomposable.
2. In this problem we investigate a fancy definition of the exterior algebra. Please review Problem 2 of Exercises 2.5. Let  $V$  be a vector space, not necessarily finite-dimensional, and define  $V^{\otimes k} = V \otimes \cdots \otimes V$  ( $k$ -fold tensor product of  $V$ ) as in Problem 2 of Exercises 2.5. (If  $V$  is finite-dimensional, this is our “usual” definition.)
- (i) Let's call an element  $f \in V^{\otimes k}$  **repeating** if  $f$  is of the form

$$f = v_1 \otimes v_2 \otimes \cdots \otimes v_k,$$

where for some  $i \neq j$ , we have  $v_i = v_j$ . Let  $R \subseteq V^{\otimes k}$  be the span of all repeating tensors. Prove that  $R$  is a left and right ideal in  $V^{\otimes k}$ .

- (ii) We define  $\wedge^k V$  as the quotient space

$$\wedge^k V := V^{\otimes k} / R.$$

Define multiplication by

$$(2.74) \quad \wedge^j V \times \wedge^k V \ni (v, w) \mapsto [x \otimes y] \in \wedge^{j+k} V,$$

where  $v = [x] \in \wedge^j V$  and  $w = [y] \in \wedge^k V$ . Prove that this multiplication is well-defined; that is, is independent of the choice of representatives  $x$  and  $y$  of the equivalence classes  $v, w$ . We denote the element  $[x \otimes y]$  by  $v \wedge w$ .

- (iii) So far we have described  $\wedge^k V$  and the wedge product for  $V$  possibly infinite-dimensional. Assume now that  $V$  is finite-dimensional. Let's change gear and consider  $\wedge^k V$  as defined in this section, that is, as alternating maps on  $(V^*)^k$ . We shall prove that  $\wedge^k V \cong V^{\otimes k} / R$ , so the above definition is consistent with what we did. To this end, first prove that the alternating operator

$$A : V^{\otimes k} \rightarrow \wedge^k V,$$

defined in (2.61), vanishes on  $R$ . (Here,  $A$  depends on  $k$ , but we omit this fact for notational simplicity.) Thus,  $A$  descends to a map on the quotient

$$A : V^{\otimes k} / R \rightarrow \wedge^k V.$$

Prove that this map is an isomorphism of vector spaces. Suggestion: Let  $v_1, \dots, v_n$  be a basis for  $V$ . Prove that the set of all equivalence classes of the form

$$\{[v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}] \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

spans  $V^{\otimes k} / R$ . Next, show that this set is linearly independent by observing that the image of this set under  $A$  in  $\wedge^k V$  is linearly independent.

- (iv) Prove that  $A : V^{\otimes k} / R \rightarrow \wedge^k V$  preserves the wedge product:

$$A(v \wedge w) = Av \wedge Aw,$$

where  $v \in V^{\otimes j} / R$  and  $w \in V^{\otimes k} / R$ . Here,  $v \wedge w$  is defined by (2.74) while  $Av \wedge Aw$  is defined by (2.62). Therefore, it is safe to identify  $\wedge^k V$  with  $V^{\otimes k} / R$ .

3. (**Universal mapping property**) Let  $V$  be a vector space, not necessarily finite-dimensional. Let  $\widehat{V}$  be a vector space and let  $\varphi : V^k \rightarrow \widehat{V}$  be an alternating map (that is, it's multi-linear and it switches sign whenever two of its arguments are switched). Suppose that  $(\widehat{V}, \varphi)$  has the following property: If  $f : V^k \rightarrow W$  is an alternating map into some vector space  $W$ , then there exists a unique *linear* map  $\tilde{f} : \widehat{V} \rightarrow W$  such that the following diagram commutes:

$$\begin{array}{ccc} V^k & \xrightarrow{\varphi} & \widehat{V} \\ & \searrow f & \downarrow \tilde{f} \\ & & W \end{array}, \quad \text{that is, } f = \tilde{f} \circ \varphi.$$

The function  $f$  is said to factor through  $\varphi$ . Thus, we can see that  $(\widehat{V}, \varphi)$  produces a linear map  $\tilde{f}$  given an alternating map  $f$ . The pair  $(\widehat{V}, \varphi)$  is said to have the universal mapping property, or is universal, for alternating maps on  $V^k$ .

- (i) Prove that if  $(\widehat{V}, \varphi)$  and  $(\widehat{V}', \varphi')$  both have the universal mapping property for alternating maps on  $V^k$ , then there is an isomorphism  $\psi : \widehat{V} \rightarrow \widehat{V}'$  such that  $\varphi' = \psi \circ \varphi$ . Congratulations, you just did some “generalized abstract nonsense”!
- (ii) Let  $\varphi : V^k \rightarrow \wedge^k V := V^{\otimes k}/R$  be the map  $(v_1, \dots, v_k) \mapsto [v_1 \otimes \cdots \otimes v_k]$ . Prove that  $(\wedge^k V, \varphi)$  has the universal mapping property for alternating maps on  $V^k$ .

In this sense, the wedge product  $\wedge^k V$  “solves” the **universal mapping problem** for alternating maps on  $V^k$ .

### Appendix: A review of permutations

In this short appendix we review some facts concerning permutations used in the previous section. Permutations are usually studied in an abstract algebra course, but just in case you’re rusty on this subject, we shall prove some of the basic properties of permutations here.

• **Permutations and transpositions.** A **permutation** on a set  $A$  is another name for a bijection on  $A$ . The group of all permutations on a finite set  $A$  elements is called the **symmetric group** on  $A$  and is sometimes denoted by  $S(A)$ . For concreteness, throughout this section we shall consider the set  $A = \{1, 2, \dots, n\}$ , in which case we denote  $S(A)$  by  $S_n$ . Our first observation is that  $S_n$  has  $n!$  elements. To see this, note that a permutation can map 1 to  $n$  numbers. It can map 2 to  $n - 1$  numbers (because it can’t map 2 to the same number that it mapped 1 to). Similarly, it can map 3 to  $n - 2$  numbers, and so on. Finally, it can map  $n$  to only 1 number. Multiplying, we conclude that  $S_n$  has  $n \cdot (n - 1) \cdots 2 \cdot 1 = n!$  elements. An important type of permutation is called a **transposition**, which switches exactly two distinct elements of  $\{1, 2, \dots, n\}$  leaving all others alone. If  $a$  and  $b$  are the switched elements, we denote the transposition by  $(a\ b)$  or  $(b\ a)$ ; the order doesn’t matter. A crucial property of transpositions is that they form the building blocks of all permutations.

PROPOSITION 2.36. *Any permutation is a product of transpositions.*

PROOF. Let  $\sigma \in S_n$ . Let  $a_1 = \sigma(1)$  and define a permutation  $\sigma_1$  by

$$\sigma_1 := (1\ a_1)\sigma.$$

Then  $\sigma_1$  is a permutation and  $\sigma_1(1) = (1\ a_1)\sigma(1) = 1$ .

Define  $a_2 = \sigma_1(2)$  and define

$$\sigma_2 := (2\ a_2)\sigma_1 = (2\ a_2)(1\ a_1)\sigma.$$

Note that  $a_2 \neq 1$  (because  $\sigma_1(1) = 1$  so  $\sigma_1(2)$  cannot equal 1), so  $\sigma_2(1) = (2\ a_2)\sigma_1(1) = (2\ a_2)(1) = 1$ . Also,  $\sigma_2(2) = (2\ a_2)\sigma_1(2) = 2$ . Thus,  $\sigma_2(1) = 1$  and  $\sigma_2(2) = 2$ .

Similarly, defining  $a_3 = \sigma_2(3)$ , the permutation

$$\sigma_3 := (3\ a_3)\sigma_2 = (3\ a_3)(2\ a_2)(1\ a_1)\sigma$$

satisfies  $\sigma_3(1) = 1$ ,  $\sigma_3(2) = 2$ , and  $\sigma_3(3) = 3$ . Continuing this process by induction  $n - 3$  more times, we eventually arrive at a permutation

$$\sigma_n = (n\ a_n) \cdots (3\ a_3)(2\ a_2)(1\ a_1)\sigma$$

that satisfies  $\sigma_n(1) = 1, \sigma_n(2) = 2, \dots, \sigma_n(n) = n$ ; that is,  $\sigma_n = i$ :

$$i = (n \ a_n) \cdots (3 \ a_3)(2 \ a_2)(1 \ a_1) \sigma.$$

Multiplying both sides by  $(1 \ a_1)(2 \ a_2) \cdots (n \ a_n)$ , we get our desired result,

$$(1 \ a_1)(2 \ a_2)(3 \ a_3) \cdots (n \ a_n) = \sigma,$$

□

• **The sign of a permutation.** Of course, the number of transpositions and types of transpositions in any given representation of a permutation may not be unique; for example,

$$(1 \ 2) = (3 \ 4)(3 \ 1)(3 \ 4)(4 \ 2)(1 \ 4).$$

However, although both sides have different types of transpositions and a different number of transpositions, notice that both sides contain an odd number of transpositions. It turns out that a permutation can be written either as an even number of transpositions or as an odd number of transpositions but *never* both. This is a consequence of the following theorem.

**THEOREM 2.37.** *If a permutation  $\sigma$  is written as a product of transpositions:*

$$\sigma = \tau_1 \tau_2 \cdots \tau_k = \sigma_1 \sigma_2 \cdots \sigma_{k'},$$

*then  $k \equiv k' \pmod{2}$ .*

**PROOF.** Bringing the right side of  $\tau_1 \tau_2 \cdots \tau_k = \sigma_1 \sigma_2 \cdots \sigma_{k'}$  to the left, we can write the identity permutation as a product

$$(2.75) \quad i = \tau_1 \tau_2 \cdots \tau_\ell,$$

where  $\ell = k + k'$ . We need to show that  $\ell$  is even. To do so we proceed via (strong) induction. Certainly  $\ell \neq 1$  and if  $\ell = 2$  then we are done. Suppose that if the identity can be written as a product of at most  $\ell - 1$  transpositions, then the number of transpositions is even; we shall prove the same is true for a product of  $\ell$  transpositions. To help us in our proof, consider the following identities for the product of arbitrary transpositions (assume that  $a, b, c, d$  are distinct):

$$\begin{aligned} I) & \quad (a \ b)(a \ b) = i \\ II) & \quad (a \ b)(a \ c) = (b \ c)(a \ b) \\ III) & \quad (a \ b)(c \ d) = (c \ d)(a \ b) \\ IV) & \quad (a \ b)(b \ c) = (b \ c)(a \ c). \end{aligned}$$

The point here is that for II) – IV), we can bring  $a$  to the second transposition having no  $a$  in the first.

In (2.75), let  $\tau_1 = (a \ b)$ . If I) holds for  $\tau_1 \tau_2$  in (2.75), then we can write  $i$  as a product of  $\ell - 2$  transpositions and we can apply the induction hypothesis to complete our proof. Thus, assume that one of II) – IV) holds. Then we can write  $i$  still in the form (2.75) but with a new  $\tau_1$  and  $\tau_2$ , where  $a$  appears in  $\tau_2$  and is not in  $\tau_1$ . Thus, we may assume that in (2.75),  $a$  appears in  $\tau_2$  and not in  $\tau_1$ . If I) holds for  $\tau_2 \tau_3$  in (2.75), then we can write  $i$  as a product of  $\ell - 2$  transpositions and the induction hypothesis completes our proof. Thus, assume that one of II) – IV) holds for  $\tau_2 \tau_3$ . Then we can write  $i$  again in the form (2.75), but with a new  $\tau_2$  and  $\tau_3$ , where  $a$  appears in  $\tau_3$  and is not in  $\tau_1$  nor in  $\tau_2$ . Thus, we may assume that in (2.75),  $a$  appears in  $\tau_3$  and not in  $\tau_1, \tau_2$ . Continuing this process by induction,

either I) occurs at some point, in which case we get our result via strong induction or I) never occurs, in which case we can write  $i$  in the form

$$i = \tau_1 \tau_2 \cdots \tau_{\ell-1}(a \alpha),$$

where  $a$  does not occur in any  $\tau_1, \dots, \tau_{\ell-1}$ . However, if this were the case, then we would have

$$\alpha = i(\alpha) = \tau_1 \tau_2 \cdots \tau_{\ell-1}(a \alpha)(\alpha) = \tau_1 \tau_2 \cdots \tau_{\ell-1}(a) = a,$$

a contradiction. This completes our proof.  $\square$

The **sign** of a permutation  $\sigma$ , denoted by  $\text{sgn } \sigma$ , is  $+1$  if  $\sigma$  can be written as a product of an even number of transpositions and  $-1$  if  $\sigma$  can be written as a product of an odd number of transpositions. More explicitly, if  $\sigma = \tau_1 \cdots \tau_\ell$ , then

$$\text{sgn } \sigma = (-1)^\ell$$

and this is well-defined, independent of the number of transpositions in a representation of  $\sigma$ , by Theorem 2.37. Important and easy-to-prove properties of  $\text{sgn}$  are that  $\text{sgn}$  is multiplicative and it preserves inversion:

$$\text{sgn}(\sigma \sigma') = \text{sgn } \sigma \cdot \text{sgn } \sigma' \quad , \quad \text{sgn } \sigma = \text{sgn}(\sigma^{-1}).$$

We end this appendix with an example used in the previous section.

**Example 2.35.** Define  $\rho \in S_{j+k}$  by the table

$$(2.76) \quad \begin{array}{c|c|c|c|c|c|c|c|c|c} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & j+k-1 & j+k \\ \hline 1+k & 2+k & \cdots & j-1+k & j+k & 1 & \cdots & k-1 & k \end{array}$$

Here,  $\rho$  takes each element in the first row to the corresponding element of the second row; e.g.  $\rho(1) = 1+k$  and so forth. We shall prove that  $\text{sgn } \rho = (-1)^{jk}$ . The idea to prove this is simple: We make  $\rho$  into the identity permutation by “switching” the elements “ $1+k, 2+k, \dots, j+k$ ” with “ $1, 2, \dots, k$ ” on the bottom row of  $\rho$  in (2.76) by composing  $\rho$  with transpositions. One way to do this is to first move  $(j+k)$  to the right by switching it with 1, then 2, etc., until we get it to the end; then we start moving  $j-1+k$  to the right until we move it in front of  $j+k$ ; and so on and so on. Although this is easy to state, in practice it’s a bit messy, so here are the details, with warts and all. First, consider the product of transpositions:

$$\sigma_1 := ((j+k) \ k) ((j+k) \ (k-1)) \cdots ((j+k) \ 2) ((j+k) \ 1).$$

Using the table (2.76), one can see that  $\sigma_1 \rho$  is given by the table

$$(2.77) \quad \begin{array}{c|c|c|c|c|c|c|c|c|c} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & j+k-1 & j+k \\ \hline 1+k & 2+k & \cdots & j-1+k & 1 & 2 & \cdots & k & j+k \end{array}$$

Now consider the product of transpositions:

$$\sigma_2 := ((j-1+k) \ k) ((j-1+k) \ (k-1)) \cdots ((j-1+k) \ 2) ((j-1+k) \ 1).$$

Using the table (2.77), one can see that  $\sigma_2 \sigma_1 \rho$  is given by the table

$$\begin{array}{c|c|c|c|c|c|c|c|c|c} 1 & 2 & \cdots & j-2 & j-1 & j & j+1 & \cdots & j+k-2 & j+k-1 & j+k \\ \hline 1+k & 2+k & \cdots & j-2+k & 1 & 2 & 3 & \cdots & k & j-1+k & j+k \end{array}$$

Continuing this process we eventually arrive at

$$\sigma_j \sigma_{j-1} \cdots \sigma_2 \sigma_1 \rho = i,$$

the identity permutation where each  $\sigma_\ell$  is a product of  $k$  transpositions. Taking  $\text{sgn}$  of both sides and using that  $\text{sgn}$  is multiplicative, we obtain  $(-1)^k \cdots (-1)^k \text{sgn } \rho = 1$ , where there are  $j$  occurrences of  $(-1)^k$ . Hence,  $\text{sgn } \rho = (-1)^{jk}$ .