

2.7. Vector bundles II: Tensors, forms, and the exterior derivative

The object of this section is to show that anything you can do with vector spaces (take duals, take tensor products, etc.) you can do with vector bundles. We also study the exterior derivative, one of the most important operators in this book.

SECTION OBJECTIVES: THE STUDENT WILL BE ABLE TO ...

- specify local trivializations of tensor, form, ... bundles.
- Explain the complexification of a real vector bundle.
- define the exterior derivative and state some of its properties.

2.7.1. Lightning review of vector bundles. It has been awhile since we have touched differential geometry, so let's quickly review vector bundles. Let E be a set and let $\pi : E \rightarrow M$ be a surjective map where M is a manifold. Recall that E is called a rank N \mathbb{K} -vector bundle ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) if for each $p \in M$, the set $E_p := \pi^{-1}(p)$ is a \mathbb{K} -vector space of dimension N . It's important to note that the \mathbb{K} refers only to the vector space structure of the fibers. For example, a *complex* vector bundle only refers to the fact that the fibers E_p are complex vector spaces; they don't have to do with any complex nature of the manifold M .¹⁴ We only deal with *smooth* vector bundles, which means that E is equipped with compatible local trivializations. To briefly recap, recall that if $\mathcal{U} \subseteq M$ is open, N sections e_1, \dots, e_N of $E|_{\mathcal{U}} := \pi^{-1}(\mathcal{U})$ are a trivialization of $E|_{\mathcal{U}}$, or a local trivialization of E , if for each $p \in \mathcal{U}$,

$$e_1(p), e_2(p), \dots, e_N(p) \text{ form a basis of } E_p.$$

We denote such a trivialization by a pair $(\mathcal{U}, \{e_i\})$. If $(\tilde{\mathcal{U}}, \{\tilde{e}_i\})$ is another trivialization and $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$, then we can write

$$e_j = \sum_{i=1}^N a_{ij} \tilde{e}_i$$

where $a_{ij} : \mathcal{U} \cap \tilde{\mathcal{U}} \rightarrow \mathbb{K}$ are functions. We say that the two trivializations are compatible if the functions a_{ij} are smooth. (We can also write each \tilde{e}_j in terms of the e_i 's via functions b_{ij} and if the a_{ij} 's are smooth, then so are the b_{ij} 's by e.g. Cramer's rule.) Finally, recall that a vector bundle atlas is a collection of trivializations

$$\mathcal{A} = \left\{ (\mathcal{U}_\alpha, \{e_i^\alpha\}) \right\}$$

such that the collection $\{\mathcal{U}_\alpha\}$ is a cover of M (that is, $M = \bigcup_\alpha \mathcal{U}_\alpha$) and any two trivializations in \mathcal{A} are compatible. Any such atlas can be made maximal and finally we arrive at our definition: A smooth rank N \mathbb{K} -vector bundle is a pair (E, \mathcal{A}) where E is a rank N \mathbb{K} -vector bundle and \mathcal{A} is a maximal atlas on E .

Given finite-dimensional vector spaces V, W , we know how to form the following vector spaces:

$$V^*, V \otimes W, V^{\otimes k}, V \oplus W, \text{hom}(V, W) \cong V^* \otimes W \cong W \otimes V^*, \Lambda^k V,$$

and so on. Recall that the identification $\text{hom}(V, W) \cong W \otimes V^*$ is given as follows. If $\alpha \in V^*$ and $w \in W$, then $w \otimes \alpha \in \text{hom}(V, W)$ is the map

$$(w \otimes \alpha)(v) := \alpha(v) w \text{ for all } v \in V.$$

¹⁴There is a notion of complex manifold but we won't deal with such objects in this book.

Since vector bundles are just “smoothly varying vector spaces” over a manifold, these operations “obviously” and easily produce vector bundles as shall see. In fact, the general principle is

General principle: *What you can do with vector spaces and their elements, you can most likely do with vector bundles and their sections.*

2.7.2. Operations on vector bundles. As a typical example, let us consider the tensor product of two vector bundles. Let E and F be \mathbb{K} -vector bundles over a manifold M . Define as a set,

$$E \otimes F := \bigcup_{p \in M} E_p \otimes F_p,$$

the union of the tensor products of the fibers. There is an obvious projection $\pi : E \otimes F \rightarrow M$ defined by taking an element of $E_p \otimes F_p$ to p . To make $E \otimes F$ into a smooth vector bundle we need local trivialisations. Let $(\mathcal{U}, \{e_i\})$ and $(\mathcal{U}, \{f_i\})$ be local trivialisations of E and F , respectively, over the same open set $\mathcal{U} \subseteq M$; this can always be accomplished by taking overlapping trivialisations of E and F and restricting them to the common overlap. Since for each $p \in \mathcal{U}$, $\{e_i(p)\}$ is a basis for E_p and $\{f_j(p)\}$ is a basis for F_p , by the Basis theorem 2.19 for tensor products, we know that

$$\{e_i(p) \otimes f_j(p)\} \text{ is a basis for } E_p \otimes F_p;$$

in other words,

$$(2.81) \quad \{(\mathcal{U}, \{e_i \otimes f_j\})\} \text{ is a local trivialization of } E \otimes F.$$

Let \mathcal{A} be the set of all such local trivialisations. To prove compatibility, let $\{(\mathcal{U}, \{e_i \otimes f_j\})\}$ and $\{(\tilde{\mathcal{U}}, \{\tilde{e}_i \otimes \tilde{f}_j\})\}$ be local trivialisations of $E \otimes F$ with $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$. Then,

$$e_j = \sum_{i=1}^{N_1} a_{ij} \tilde{e}_i$$

where $a_{ij} : \mathcal{U} \cap \tilde{\mathcal{U}} \rightarrow \mathbb{K}$ are smooth and

$$f_j = \sum_{i=1}^{N_2} b_{ij} \tilde{f}_i$$

where $b_{ij} : \mathcal{U} \cap \tilde{\mathcal{U}} \rightarrow \mathbb{K}$ are smooth functions. Hence,

$$e_i \otimes f_j = \sum_{k,\ell} a_{\ell i} b_{kj} \tilde{e}_k \otimes \tilde{f}_\ell$$

where $a_{\ell i} b_{kj} : \mathcal{U} \cap \tilde{\mathcal{U}} \rightarrow \mathbb{K}$ are smooth. Therefore, the trivialisations $\{(\mathcal{U}, \{e_i \otimes f_j\})\}$ and $\{(\tilde{\mathcal{U}}, \{\tilde{e}_i \otimes \tilde{f}_j\})\}$ are compatible. Completing \mathcal{A} we get a vector bundle atlas for $E \otimes F$, which shows that $E \otimes F$ is a smooth vector bundle. It's that easy!

Example 2.36. The main example is $T^*M \otimes T^*M$. Here $E = F = T^*M$ is the cotangent bundle, a real vector bundle. If (\mathcal{U}, x) is a coordinate patch on M , then we know that $\{dx_1, \dots, dx_n\}$ (where $n = \dim M$) is a trivialization of $T^*M|_{\mathcal{U}}$, hence by (2.81) we know that

$$\{dx_i \otimes dx_j \mid i, j = 1, \dots, n\}$$

is a trivialization of $(T^*M \otimes T^*M)|_{\mathcal{U}}$. By the way, another notation of $T^*M \otimes T^*M$ is $T^*M^{\otimes 2}$. Recall that, by definition, a smooth section $f \in C^\infty(M, T^*M^{\otimes 2})$ is a map $f : M \rightarrow T^*M^{\otimes 2}$ such that $f(p) \in T_p^*M^{\otimes 2}$ for all $p \in M$, and smooth means that in any local trivialization $\{dx_i \otimes dx_j\}$ of $T^*M^{\otimes 2}|_{\mathcal{U}}$, we can write

$$f = \sum_{i,j} f_{ij} dx_i \otimes dx_j,$$

where $f_{ij} : \mathcal{U} \rightarrow \mathbb{R}$ is smooth for all i, j . The most important section of $T^*M^{\otimes 2}$ is a Riemannian metric, so be discussed in Section 2.10.

Here is another example, an important example that we'll use quite often, the form bundle:

$$\Lambda^k E := \bigcup_{p \in M} \Lambda^k E_p$$

There is an obvious projection $\pi : \Lambda^k E \rightarrow M$ defined by taking an element of $\Lambda^k E_p$ to p . If $k > N$ with $N = \text{rank } E$, then $\Lambda^k E_p = \{0\}$, the zero vector space, for each $p \in M$. So, in the case that $k > N$, $\Lambda^k E$ is really $M \times \{0\}$, a trivial rank zero vector bundle. If $k = 0$, then $\Lambda^0 E_p := \mathbb{K}$ for all $p \in M$, so in this case $\Lambda^0 E = M \times \mathbb{K}$, another trivial vector bundle. So, let us assume that $1 \leq k \leq N$. To define a local trivialization of $\Lambda^k E$, let $(\mathcal{U}, \{e_i\})$ be a local trivialization of E . Since for each $p \in \mathcal{U}$, $\{e_i(p)\}$ is a basis for E_p by the Basis theorem 2.33 for k -forms, we know that

$$\{e_I(p) := e_{i_1}(p) \wedge e_{i_2}(p) \wedge \cdots \wedge e_{i_k}(p)\}$$

where $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq N$ is a basis for $\Lambda^k E_p$. Thus,

$$(2.82) \quad \{(\mathcal{U}, \{e_I\})\} \text{ is a local trivialization of } \Lambda^k E.$$

Let \mathcal{A} be the set of all such local trivializations. To prove compatibility, let $\{(\tilde{\mathcal{U}}, \{\tilde{e}_I\})\}$ and $\{(\mathcal{U}, \{e_J\})\}$ be local trivializations of $\Lambda^k E$ (defined as we did above) with $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$. We know that

$$e_j = \sum_{i=1}^{N_1} a_{ij} \tilde{e}_i$$

where $a_{ij} : \mathcal{U} \cap \tilde{\mathcal{U}} \rightarrow \mathbb{K}$ are smooth. Hence, for $1 \leq j_1 < \cdots < j_k \leq N$, we have

$$e_{j_1} \wedge \cdots \wedge e_{j_k} = \sum_{\ell_1, \dots, \ell_k=1}^N a_{\ell_1, j_1} \cdots a_{\ell_k, j_k} \tilde{e}_{\ell_1} \wedge \cdots \wedge \tilde{e}_{\ell_k}$$

where the sum on the right is over all $\ell_1 = 1, \dots, N, \dots, \ell_k = 1, \dots, N$. We can write this as

$$e_J = \sum_{1 \leq i_1 < \cdots < i_k \leq N} a_I \tilde{e}_I,$$

where

$$a_I = \sum_L \text{sgn } \sigma_{IL} a_{\ell_1, j_1} \cdots a_{\ell_k, j_k},$$

where the sum is over those k -tuples $L = (\ell_1, \dots, \ell_k)$ that are a permutation of $I = (i_1, \dots, i_k)$, and σ_{IL} is the permutation of (i_1, \dots, i_k) that takes $i_j \mapsto \ell_j$. The a_I 's are certainly smooth, being a linear combination of products of smooth functions, so it follows that $\{(\tilde{\mathcal{U}}, \{\tilde{e}_I\})\}$ and $\{(\mathcal{U}, \{e_J\})\}$ are compatible. Completing \mathcal{A} we

get a vector bundle atlas for $\Lambda^k E$, which shows that $\Lambda^k E$ is a smooth vector bundle. By Theorem 2.33 we know that $\Lambda^k E$ has rank $\binom{N}{k}$.

Example 2.37. We are mostly interested in the example $\Lambda^k T^*M$. As with the previous example, $E = T^*M$ is the cotangent bundle, a real vector bundle, so $\Lambda^k T^*M$ is also a real vector bundle. We usually denote $\Lambda^k T^*M$ by $\Lambda^k M$ or simply by Λ^k if this won't cause confusion. If (\mathcal{U}, x) is a coordinate patch on M , then we know that $\{dx_1, \dots, dx_n\}$ (where $n = \dim M$) is a trivialization of $T^*M|_{\mathcal{U}}$, hence by (2.82) we know that

$$\{dx_I \mid i = (i_1, \dots, i_k) \text{ with } 1 \leq i_1 < \dots < i_k \leq n\}$$

is a trivialization of $\Lambda^k M|_{\mathcal{U}}$. Here we assume that $1 \leq k \leq n$. By definition, a smooth section $\alpha \in C^\infty(M, \Lambda^k)$ is a map $\alpha : M \rightarrow \Lambda^k M$ such that $\alpha(p) \in \Lambda^k T_p^*M$ for all $p \in M$, and smooth means that in any local trivialization $\{dx_I\}$ of $\Lambda^k M|_{\mathcal{U}}$, we can write

$$\alpha = \sum_I a_I dx_I,$$

where $a_I : \mathcal{U} \rightarrow \mathbb{R}$ is smooth for all I .

2.7.3. The general principle. The examples $E \otimes F$ and $\Lambda^k E$ given above are in some sense typical: We can “easily” make

$$E^* := \bigcup_{p \in M} E_p^*, \quad E \oplus F := \bigcup_{p \in M} E_p \oplus F_p,$$

$$E^{\otimes k} := \bigcup_{p \in M} (E_p)^{\otimes k}, \quad \text{hom}(E, F) := \bigcup_{p \in M} \text{hom}(E_p, F_p),$$

into vector bundles by just defining their local trivializations from the local trivializations of E and F in the “obvious” way. Since these constructions are helpful to the reader and are morally similar to the tensor product and form examples above, we shall leave their verification to the reader; see Problem 1. However, we make two remarks. First, for the hom bundle $\text{hom}(E, F)$, one can show that

$$\text{hom}(E, F) \cong E^* \otimes F \cong F \otimes E^*$$

where \cong means bundle isomorphic (which of course is true for vector spaces). Second, it is useful to know certain facts concerning local trivializations (which are in analogy to our various basis theorems). We list them in the following proposition.

PROPOSITION 2.38. *Let E and F be \mathbb{K} vector bundles over a manifold M . Let $\{e_i\}$ and $\{f_j\}$ be local trivializations of E and F , respectively, over an open set \mathcal{U} of M . Then,*

- (i) $\{e_i^*\}$ is a local trivialization of E^* over \mathcal{U} . Here, the section $e_i^* \in C^\infty(\mathcal{U}, E^*)$ is the section satisfying $e_i^*(e_j) = \delta_{ij}$ for all j .
- (ii) $\{e_i \otimes f_j\}$, $\{f_i \otimes e_j^*\}$, and $\{e_i, f_j\}$ are local trivialization of $E \otimes F$, $F \otimes E^* \cong \text{hom}(E, F)$, and $E \oplus F$, respectively, over \mathcal{U} .
- (iii) $\{e_i \otimes e_j \otimes \dots \otimes e_l\}$ (all possible k -fold tensor products of $\{e_i\}$) is a local trivialization of $E^{\otimes k}$ over \mathcal{U} .
- (iv) $\{e_I \mid I = (i_1, \dots, i_k) \text{ with } 1 \leq i_1 < \dots < i_k \leq \text{rank } E\}$ is a local trivialization of $\Lambda^k E$ over \mathcal{U} .

In Problem 1 to prove this proposition if you like. We now show how to generate smooth sections of the vector bundles we have just constructed. Let $e \in C^\infty(X, E)$, $f \in C^\infty(M, F)$, $\alpha \in C^\infty(M, E^*)$, and let $T \in C^\infty(X, \text{hom}(E, F))$. We define sections

$$e \otimes f \in C^\infty(M, E \otimes F), \quad e \oplus f \in C^\infty(X, E \oplus F),$$

$$\alpha(e) \in C^\infty(M, \mathbb{K}), \quad T(e) \in C^\infty(M, F)$$

pointwise as follows: For each $p \in M$,

$$(e \otimes f)(p) := e(p) \otimes f(p) \in E_p \otimes F_p;$$

$$(e \oplus f)(p) := e(p) \oplus f(p) \in E_p \oplus F_p;$$

$$\alpha(e)(p) := \alpha(p)(e(p)) \in \mathbb{K};$$

and

$$T(e)(p) := T(p)(e(p)) \in F_p.$$

The proofs that the sections just defined are smooth (as claimed) are really elementary so we leave their verification to the reader. Because $\text{hom}(E, F) \cong F \otimes E^*$ observe that

$$f \otimes \alpha \in C^\infty(M, \text{hom}(E, F)).$$

Here is another fact: If $e_1 \in C^\infty(M, \Lambda^j E)$ and $e_2 \in C^\infty(M, \Lambda^k E)$, then (no surprise!)

$$e_1 \wedge e_2 \in C^\infty(M, \Lambda^{j+k} E),$$

where $e_1 \wedge e_2$ is defined pointwise: For each $p \in M$,

$$(e_1 \wedge e_2)(p) := e_1(p) \wedge e_2(p) \in \Lambda^{j+k} E_p.$$

Of course, we can go on just for pages and pages with examples, but let us suffice to repeat the general principle:

General principle: *What you can do with vector spaces and their elements, you can most likely do with vector bundles and their sections.*

We now consider operations a little different from what we've studied.

2.7.4. More operations. An operation we can perform on vector bundles but not on a vector space is the pull-back operation. Let E be a \mathbb{K} -vector bundle over a manifold N and let $f : M \rightarrow N$ be a smooth map. Then the **pull-back bundle** f^*E is the vector bundle

$$f^*E := \bigcup_{p \in M} (f^*E)_p,$$

where the fibers are by definition

$$(f^*E)_p := E_{f(p)} \quad \text{for all } p \in M.$$

In other words, we pull back the fiber over $f(p)$ to the fiber over p . If you're into diagrams, here is a diagram representing this situation:

$$\begin{array}{ccc}
 f^*(E) & \xrightarrow{\text{Id}} & E \\
 \downarrow \pi & & \downarrow \pi \\
 M & \xrightarrow{f} & N
 \end{array}
 \quad \text{that is,} \quad
 \begin{array}{ccc}
 f^*(E)_p := E_{f(p)} & \xrightarrow{\text{Id}} & E_{f(p)} \\
 \downarrow \pi & & \downarrow \pi \\
 p & \xrightarrow{f} & f(p)
 \end{array}
 .$$

In Problem 1 you can verify that f^*E satisfies the definition of being a vector bundle. This pull back operation will show up in Section 3.2 when we define the symbols of differential operators.

Another operation that we need to discuss is the “box product”. Let E and F be \mathbb{K} -vector bundles over M . Then we define $E \boxtimes F$ as the vector bundle over M^2 with underlying set

$$E \boxtimes F := \bigcup_{(p,q) \in M \times M} (E \boxtimes F)_{(p,q)}$$

where the fibers are by definition

$$(E \boxtimes F)_{(p,q)} := E_p \otimes F_q.$$

Note that if $F = E^*$, then $E_p \otimes E_q^* \cong \text{hom}(E_q, E_p)$. Because of this fact, we will sometimes denote $E \boxtimes E^*$ by $\text{Hom}(E)$. This “big hom bundle” will show up in Section 4.1 of Chapter 4 when we discuss integral operators and outline the heat kernel proof of the index theorem.

There is one more operation we need, the complexification of a vector bundle. The vector bundles that we know best are the bundles TM , T^*M , the tensor product bundles $T^*M^{\otimes k}$, and the form bundles $\Lambda^k M$. These vector bundles are all *real* vector bundles; that is, the fibers are real vector spaces. However, the Atiyah-Singer index theorem in its most general form, at least for this book, involves *complex* vector bundles; that is, the fibers are complex vector spaces. This is only because complex numbers is complete while the real numbers is not (so, for example, any real or complex matrix has a complex eigenvalue — not so for real eigenvalues). If we limited ourselves only to real numbers it would only complicate proofs and sometimes make them impossible. For this reason, we shall take our real vector bundles and make them complex via a very simple procedure.

Let’s start with vector spaces. Consider the following lemma, which we highly recommend you to prove if it’s not immediately obvious:

LEMMA 2.39. *If W is an N -dimensional complex vector space, then W is also a $2N$ -dimensional real vector space. (Since $\mathbb{R} \subseteq \mathbb{C}$, the action of \mathbb{C} on W restricts to an action of \mathbb{R} on W .) Moreover, a set of N vectors $\{w_1, \dots, w_N\}$ is a basis of W considered as a complex vector space if and only if $\{w_1, \dots, w_N, iw_1, \dots, iw_N\}$ is a basis of W considered as a real vector space.*

The simplest example of this is $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$, which is a one-dimensional complex vector space (with basis $\{1\}$ say), but we can also consider \mathbb{C} as a two-dimensional real vector space (with basis $\{1, i\}$). Let V be an n -dimensional real vector space. Considering \mathbb{C} as a two-dimensional real vector space it follows that

$$\boxed{\mathbb{C}V := \mathbb{C} \otimes_{\mathbb{R}} V}$$

is a $2n$ -dimensional real vector space; the subscript \mathbb{R} , which we usually omit, is simply to emphasize that we are considering \mathbb{C} as a real vector space so we are taking the tensor product of two real vector spaces. It turns out that $\mathbb{C}V$ also has the structure of a *complex* vector space. The action of \mathbb{C} on $\mathbb{C}V$ given as follows: If $a \in \mathbb{C}$, we define

$$(2.83) \quad a \cdot (b \otimes v) := (ab) \otimes v \in \mathbb{C}V, \quad \text{for all } b \in \mathbb{C}, v \in V$$

and extending by linearity to define $a \cdot \xi$ for all elements $\xi \in \mathbb{C}V$. You can also define this action as follows: We have a bilinear map

$$\mathbb{C} \times V \rightarrow \mathbb{C}V := \mathbb{C} \otimes V \quad \text{defined by } (b, v) \mapsto (ab) \otimes v.$$

This map is real multi-linear, so by the universal property of the tensor product we get a linear map

$$\mathbb{C} \otimes V = \mathbb{C}V \rightarrow \mathbb{C}V \quad \text{such that } b \otimes v \mapsto (ab) \otimes v;$$

this is the map (2.83). Any case, observe that as $\{1, i\}$ and $\{v_j\}$ are bases for the vector spaces \mathbb{C} (considered as a real vector space) and V , respectively, so by our basis theorem for tensor products, we know that

$$\{1 \otimes v_1, \dots, 1 \otimes v_n, i \otimes v_1, \dots, i \otimes v_n\}$$

is a basis for the $2n$ -dimensional real vector space $\mathbb{C}V = \mathbb{C} \otimes_{\mathbb{R}} V$. By definition of the action of \mathbb{C} on $\mathbb{C}V$, this basis can be written as

$$\{1 \otimes v_1, \dots, 1 \otimes v_n, i(1 \otimes v_1), \dots, i(1 \otimes v_n)\},$$

so by Lemma 2.39, $\{1 \otimes v_1, \dots, 1 \otimes v_n\}$ is a basis of $\mathbb{C}V$ considered as a complex vector space. We shall denote an element $1 \otimes v \in \mathbb{C}V = \mathbb{C} \otimes_{\mathbb{R}} V$ by v , so we can simply say that $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{C}V$ considered as a complex vector space. We call $\mathbb{C}V$ the **complexification** of V .

PROPOSITION 2.40. *If $\{v_1, \dots, v_n\}$ is a basis for a real vector space V , then $\{v_1, \dots, v_n\}$ is a basis for the complex vector space $\mathbb{C}V$. In particular, $\mathbb{C}V$ is an n -dimensional complex vector space.*

We now move to vector bundles, so let E be a rank N real vector bundle over M . We define

$$\mathbb{C}E := \bigcup_{p \in M} \mathbb{C}E_p,$$

where $\mathbb{C}E_p$ is the complexification of the real vector space E_p . By our discussion above, the fibers of $\mathbb{C}E$ are N -dimensional complex vector spaces. If $\{e_j\}$ is a local basis of the real vector bundle E , then $\{e_j\}$ (really $\{1 \otimes e_j\}$) is a local basis of the complex vector space $\mathbb{C}E$. The complex vector bundle $\mathbb{C}E$ is called the **complexification** of E .

2.7.5. The exterior derivative. Before jumping in to the exterior derivative, we continue our discussion of grad, curl, div started in the prelude to Section 2.5. To briefly review, recall that for \mathbb{R}^3 , the exterior derivative is the operator

$$d = \partial_x dx + \partial_y dy + \partial_z dz,$$

which when applied to a function is just the differential of the function. Let α be a one-form:

$$\alpha = P dx + Q dy + R dz \in C^\infty(\mathbb{R}^3, T^*\mathbb{R}^3) = C^\infty(\mathbb{R}^3, \Lambda^1).$$

Then we saw that $d\alpha \in C^\infty(\mathbb{R}^3, \Lambda^2)$ is the two-form

$$\begin{aligned} d\alpha &= d(P dx + Q dy + R dz) \\ &= (dP) \wedge dx + (dQ) \wedge dy + (dR) \wedge dz \\ &= (\partial_y R - \partial_z Q) dy \wedge dz + (\partial_z P - \partial_x R) dz \wedge dx + (\partial_x Q - \partial_y P) dx \wedge dy \\ &= \text{“curl” of } \alpha. \end{aligned}$$

We also saw that d applied to a two-form

$$\alpha = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \in C^\infty(\mathbb{R}^3, \Lambda^2)$$

is the three-form $d\alpha \in C^\infty(\mathbb{R}^3, \Lambda^3)$ defined by

$$\begin{aligned} d\alpha &= d(P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy) \\ &= (dP) \wedge dy \wedge dz + (dQ) \wedge dz \wedge dx + (dR) \wedge dx \wedge dy \\ &= (\partial_x P + \partial_y Q + \partial_z R) dx \wedge dy \wedge dz \\ &= \text{“div” of } \alpha. \end{aligned}$$

For \mathbb{R}^3 , the exterior derivative contains the gradient, curl, and divergence. We now extend the exterior derivative for \mathbb{R}^3 to any manifold.

For an n -dimensional manifold M , we define

$$\Lambda M := \Lambda^0 M \oplus \Lambda^1 M \oplus \cdots \oplus \Lambda^n M,$$

the total form bundle. We shall define a linear map

$$d : C^\infty(M, \Lambda) \rightarrow C^\infty(M, \Lambda),$$

such that for each k ,

$$d : C^\infty(M, \Lambda^k) \rightarrow C^\infty(M, \Lambda^{k+1}).$$

To this end, let $\alpha \in C^\infty(M, \Lambda^k)$; we shall define $d\alpha \in C^\infty(M, \Lambda^{k+1})$. To do so, given a coordinate patch (\mathcal{U}, x) on M , we shall define

$$(d\alpha)|_{\mathcal{U}} \in C^\infty(\mathcal{U}, \Lambda^{k+1}).$$

Since coordinate patches cover all of M we have actually defined a global object $d\alpha \in C^\infty(M, \Lambda^{k+1})$. Now to define $(d\alpha)|_{\mathcal{U}}$, we first write

$$\alpha = \sum a_I dx_I,$$

where $a_I \in C^\infty(\mathcal{U}, \mathbb{R})$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n = \dim M$. By analogy with our definition for \mathbb{R}^3 , we define

$$(2.84) \quad (d\alpha)|_{\mathcal{U}} := \sum da_I \wedge dx_I \quad \text{for } \alpha = \sum a_I dx_I.$$

Since $a_I \in C^\infty(\mathcal{U}, \mathbb{R})$, $da_I \in C^\infty(\mathcal{U}, \Lambda^1)$ and hence $(d\alpha)|_{\mathcal{U}} \in C^\infty(\mathcal{U}, \Lambda^{k+1})$. As it stands, the definition (2.84) looks coordinate dependent, that is, how do we know that if $(\tilde{\mathcal{U}}, y)$ is another coordinate patch with $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$, then writing $\alpha = \sum b_I dy_I$ with respect to the coordinates $(\tilde{\mathcal{U}}, y)$, we have

$$\sum da_I \wedge dx_I = \sum db_I \wedge dy_I \quad \text{over } \mathcal{U} \cap \tilde{\mathcal{U}}?$$

If this equality is not true, then the definition (2.84) makes no sense! However, since d seems to be a God-given operator, it would be very surprising if the definition (2.84) didn't make sense. To prove well-definedness, consider the following lemma.

LEMMA 2.41. *Fix a coordinate patch (\mathcal{U}, x) on M . Let $\mathcal{V} \subseteq \mathcal{U}$ be open and define, for any $k = 0, 1, 2, \dots$*

$$d : C^\infty(\mathcal{V}, \Lambda^k) \rightarrow C^\infty(\mathcal{V}, \Lambda^{k+1})$$

by the formula (2.84) for any $\alpha \in C^\infty(\mathcal{V}, \Lambda^k)$. Then d is linear, d is an anti-derivation in the sense that if $\alpha \in C^\infty(\mathcal{V}, \Lambda^k)$ and $\beta \in C^\infty(\mathcal{V}, \Lambda)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta,$$

and finally, $d(d\alpha) = 0$, that is, “ $d^2 = 0$ ”.

PROOF. It is obvious that d is linear. To prove the rest of the properties, we first observe that if i_1, \dots, i_k are any k integers between 1 and n , in any order (not necessarily increasing), then for any function $f \in C^\infty(\mathcal{V}, \mathbb{R})$, we have

$$(2.85) \quad d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

To see this, observe that if $k > n$ or there are any repeated integers in i_1, \dots, i_k , then both sides are zero. On the other hand, if $1 \leq k \leq n$ and there are no repeats in i_1, \dots, i_k , then we can reorder them in increasing order: $1 \leq j_1 < \cdots < j_k \leq n$, in which case

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} = c dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

where $c = \pm 1$. Hence,

$$\begin{aligned} d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) &= d(cf dx_{j_1} \wedge \cdots \wedge dx_{j_k}) := d(cf) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\ &= c df \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\ &= df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

We now prove our lemma. By multi-linearity of wedge, to prove the anti-derivation property we may assume that α and β are multiples of basis elements:

$$\alpha = a dx_I \in C^\infty(\mathcal{V}, \Lambda^k) \quad \text{and} \quad \beta = b dx_J \in C^\infty(\mathcal{V}, \Lambda^\ell),$$

where $a, b \in C^\infty(\mathcal{V}, \mathbb{R})$. Then,

$$\alpha \wedge \beta = a b dx_I \wedge dx_J.$$

Therefore by (2.85), the (easily proved) fact that $d(ab) = b da + a db$, and our properties of wedge, we have

$$\begin{aligned} d(\alpha \wedge \beta) &= d(ab) \wedge dx_I \wedge dx_J = b da \wedge dx_I \wedge dx_J + a db \wedge dx_I \wedge dx_J \\ &= b da \wedge dx_I \wedge dx_J + a db \wedge dx_I \wedge dx_J \\ &= da \wedge dx_I \wedge (b dx_J) + a(-1)^k dx_I \wedge db \wedge dx_J \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \end{aligned}$$

Finally, to prove that $d(d\alpha) = 0$ by linearity we may assume that α is a multiple of a basis element: $\alpha = f dx_I$ where $f \in C^\infty(\mathcal{V}, \mathbb{R})$. Then,

$$d\alpha = df \wedge dx_I = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_I.$$

Hence, by (2.85),

$$(2.86) \quad \begin{aligned} d(d\alpha) &= \sum_{i=1}^n \sum_{i=1}^n d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i \wedge dx_I \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I. \end{aligned}$$

By anti-commutativity of wedge and commutativity of mixed partial derivatives, we know that

$$\begin{aligned} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i + \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j &= -\frac{\partial^2 f}{\partial x_j \partial x_i} dx_i \wedge dx_j + \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \\ &= \left(-\frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dx_i \wedge dx_j = 0. \end{aligned}$$

Therefore, the terms in (2.86) cancel in pairs, and hence $d^2\alpha = 0$. \square

Using this lemma we can now prove the following theorem.

THEOREM 2.42 (The exterior derivative). *For any manifold M there is a unique linear map*

$$d : C^\infty(M, \Lambda) \rightarrow C^\infty(M, \Lambda)$$

having the following properties:

- (i) For each k , $d : C^\infty(M, \Lambda^k) \rightarrow C^\infty(M, \Lambda^{k+1})$.
- (ii) $d(f) = df$, the usual differential, for $f \in C^\infty(M, \mathbb{R}) = C^\infty(M, \Lambda^0)$.
- (iii) If $\alpha \in C^\infty(M, \Lambda^k)$ and $\beta \in C^\infty(M, \Lambda)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

- (iv) $d^2 = 0$.

PROOF. We shall prove existence and leave uniqueness (which we don't need) for your enjoyment. For existence we just have to prove that the definition (2.84) is coordinate independent, for then the rest of the properties follow from Lemma 2.41. To prove that $(d\alpha)|_{\mathcal{U}}$ is well-defined, let $(\tilde{\mathcal{U}}, y)$ be another coordinate patch with $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$. Then writing $\alpha = \sum b_I dy_I$ with respect to the coordinates $(\tilde{\mathcal{U}}, y)$, we have to prove that

$$\sum da_I \wedge dx_I = \sum db_I \wedge dy_I \quad \text{over } \mathcal{U} \cap \tilde{\mathcal{U}}.$$

To this end, let us define d via the formula (2.84) with respect to the (\mathcal{U}, x) coordinate system. Then $\mathcal{V} := \mathcal{U} \cap \tilde{\mathcal{U}} \subseteq \mathcal{U}$, so by Lemma 2.41 we have

$$\begin{aligned} \sum_I da_I \wedge dx_I &=: d\left(\sum_I a_I dx_I\right) = d\alpha = d\left(\sum_I b_I dy_I\right) \\ &= \sum_I db_I \wedge dy_I + b_I d(dy_I) \\ &= \sum_I db_I \wedge dy_I, \end{aligned}$$

where we used the anti-derivation property

$$d(dy_I) = d(dy_{i_1} \wedge \cdots \wedge dy_{i_k}) = \sum_{j=1}^k (-1)^{j-1} dy_{i_1} \wedge \cdots \wedge d^2 y_{i_j} \wedge \cdots \wedge dy_{i_k} = 0$$

because $d^2 = 0$. Therefore $\sum da_I \wedge dx_I = \sum db_I \wedge dy_I$ over $\mathcal{U} \cap \tilde{\mathcal{U}}$ and our proof is complete. \square

In particular, if $\alpha \in C^\infty(M, \Lambda^1)$, then $d\alpha \in C^\infty(M, \Lambda^2)$, and hence given any two vector fields $v, w \in C^\infty(M, TM)$, we have $d\alpha(v, w) \in C^\infty(M, \mathbb{R})$. In the following theorem we give a useful formula for $d\alpha(v, w)$.

THEOREM 2.43. *For any one-form $\alpha \in C^\infty(M, \Lambda^1)$ and vector fields $v, w \in C^\infty(M, TM)$, we have*

$$(d\alpha)(v, w) = v(\alpha(w)) - w(\alpha(v)) - \alpha([v, w]).$$

PROOF. By writing $\alpha = \sum f_i dx_i$ in local coordinates and using linearity, it suffices to prove our theorem for just one term $f_i dx_i$. Since x_i is just a function we might as well assume that $\alpha = f dg$ where f and g are smooth functions. Then $d\alpha = df \wedge dg$ and hence,

$$(d\alpha)(v, w) = df(v) dg(w) - dg(v) df(w) = (vf)(wg) - (vg)(wf).$$

Adding and subtracting $f \cdot (vwg)$ and $f \cdot (wv g)$, we see that

$$\begin{aligned} (d\alpha)(v, w) &= (vf)(wg) + f \cdot (vwg) - f \cdot (vwg) \\ &\quad - (vg)(wf) - f \cdot (wv g) + f \cdot (wv g) \\ &= v(f(wg)) - w(f(vg)) - f[v, w]g \\ &= v(f dg(w)) - w(f dg(v)) - f dg([v, w]) \\ &= v(\alpha(w)) - w(\alpha(v)) - \alpha([v, w]). \end{aligned}$$

□

In fact, there is a more general and complicated formula, but its proof is much more involved. We'll never need to use the following formula, so we leave it as an exercise if you wish:

THEOREM 2.44. *For any k -form $\alpha \in C^\infty(M, \Lambda^k)$ and vector fields $v_0, \dots, v_k \in C^\infty(M, TM)$, we have*

$$\begin{aligned} (d\alpha)(v_0, v_1, \dots, v_k) &= \sum_{j=0}^k (-1)^j v_j \alpha(v_0, \dots, \hat{v}_j, \dots, v_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k), \end{aligned}$$

where the hat “ $\hat{}$ ” means to omit the corresponding term.

For example, if α is a 2-form, then

$$\begin{aligned} d\alpha(u, v, w) &= u \alpha(v, w) + v \alpha(w, u) + w \alpha(u, v) \\ &\quad - \alpha([u, v], w) - \alpha([v, w], u) - \alpha([w, u], v). \end{aligned}$$

We don't recommend using the above theorem for $k \geq 4$ because it's just too complicated.

EXERCISES 2.7.

1. Show carefully that at least one of the following sets are (smooth) vector bundles.
 - (i) The dual bundle E^*
 - (ii) The direct sum bundle $E \oplus F$.
 - (iii) The pull-back bundle f^*E where $f : M \rightarrow N$ is a smooth map between manifolds and $E \rightarrow N$ is a vector bundle.
 - (iv) The product bundle $E \boxtimes F$.
 - (v) Prove at least one of the statements in Proposition 2.38.
2. For any open set $\mathcal{U} \subseteq \mathbb{R}^3$, functions $f, g \in C^\infty(\mathcal{U}, \mathbb{R})$, and vector field $v \in C^\infty(\mathcal{U}, T\mathcal{U})$, by relating the operators of vector calculus to d , prove the following:

- (i) $\nabla(fg) = (\nabla f)g + f(\nabla g)$
 - (ii) $\operatorname{curl}(fv) = \nabla f \times v + f \operatorname{curl} v$.
 - (iii) $\operatorname{curl} \nabla f = 0$
 - (iv) $\operatorname{div} \operatorname{curl} v = 0$.
3. (**Poincaré's lemma**) For a manifold M , a form $\alpha \in C^\infty(M, \Lambda^k)$ is said to be **closed** if $d\alpha = 0$ and **exact** if $\alpha = d\beta$ for some $\beta \in C^\infty(M, \Lambda^{k-1})$.

- (i) Prove that every exact form is also closed. Because of topological reasons it is not true, in general, that every closed form is exact; in fact, this is our main subject of study in the next two sections! However for simple manifolds this is true; for example, \mathbb{R}^n :
- (ii) Prove that every closed form on \mathbb{R}^n is exact (called **Poincaré's lemma**) as follows. Prove it when $k = 0$. Henceforth we assume $k \geq 1$ and let $\alpha \in C^\infty(\mathbb{R}^n, \Lambda^k)$ such that $d\alpha = 0$. Write $\alpha = \sum_I a_I(x) dx_I$ with respect to the usual basis for forms and define

$$\beta := - \sum_I \sum_{j=1}^k \left((-1)^{j-1} \int_0^1 a_I(tx) t^{k-1} dt \right) x_{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k},$$

where the hat means to omit the corresponding entry. Show that $\alpha = d\beta$.

By the way, in Theorem 2.53 of Section 2.9 we shall prove a more general theorem, also called Poincaré's lemma, which states that any closed form is exact on any manifold that is homotopy equivalent to a point.¹⁵

4. Using the previous exercise, prove the following "well-known" facts from elementary calculus: Over \mathbb{R}^3 , we have
- (i) $\nabla f = 0$ implies f is constant
 - (ii) $\operatorname{curl} v = 0$ implies there is a function such that $v = \nabla f$.
 - (iii) $\operatorname{div} v = 0$ implies there is a vector field w such that $v = \operatorname{curl} w$.
 - (iv) For any function f there is a vector field v such that $\operatorname{div} v = f$.

¹⁵The formula for β such that $\alpha = d\beta$ above didn't appear out of nothing: We just carefully studied the proof of Theorem 2.50 of Section 2.9 with the homotopy $H : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ defined by $H(x, t) = tx$, which contracts all of \mathbb{R}^n to the point 0, and from the proof of Theorem 2.50 we derived the formula for β above.