(ii) Prove that every closed form on $\mathbb{R}^n$ is exact (called **Poincaré's lemma**) as follows. Prove it when $k = 0$. Henceforth we assume $k \geq 1$ and let $\alpha \in C^\infty(\mathbb{R}^n, \Lambda^k)$ such that $d\alpha = 0$. Write $\alpha = \sum_I a_I(x)\, dx_I$ with respect to the usual basis for forms and define

$$\beta := -\sum_I \sum_{j=1}^k \left((-1)^{j-1} \int_0^1 a_I(tx)\, t^{k-1}\, dt \right) x_{i_j}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k},$$

where the hat means to omit the corresponding entry. Show that $\alpha = d\beta$.

By the way, in Theorem 2.53 of Section 2.9 we shall prove a more general theorem, also called Poincaré's lemma, which states that any closed form is exact on any manifold that is homotopy equivalent to a point.[15]

4. Prove the following "well-known" facts from elementary calculus (you may use the previous exercise for some of them): Over $\mathbb{R}^3$, we have
   (i) $\nabla f = 0$ implies f is constant.
   (ii) $\operatorname{curl} v = 0$ implies there is a function such that $v = \nabla f$.
   (iii) $\operatorname{div} v = 0$ implies there is a vector field w such that $v = \operatorname{curl} w$.
   (iv) For any function f there is a vector field v such that $\operatorname{div} v = f$.

2.8. The de Rham cohomology I: Basic definitions and properties

In this section we study the de Rham cohomology spaces of a manifold, which give a reasonable "measurement" of the non-trivial topological aspects of the manifold.

SECTION OBJECTIVES: THE STUDENT WILL BE ABLE TO ...

- explain the de Rham cohomology spaces to calculus students.
- compute the de Rham spaces for simple regions in $\mathbb{R}^2$.
- define the pushforward and pullback.

2.8.1. Common myths from elementary calculus. Here are some common myths that one might pick up from elementary calculus (not from the instructor but from not paying close enough attention in class!): For a vector field v on a region $\mathcal{U} \subseteq \mathbb{R}^3$,

$$\operatorname{curl} v = 0 \iff v = \nabla\varphi \text{ for some } \varphi \in C^\infty(\mathcal{U}, \mathbb{R}),$$

$$\operatorname{div} v = 0 \iff v = \operatorname{curl} w \text{ for some vector field } w.$$

It turns out that the direction "$\Longleftarrow$" (sufficiency) is always true — this is just the fact that partial derivatives commute — but there are *topological obstructions* for the direction "$\Longrightarrow$" (necessity) to be valid. For example, let's consider the first one (because it's easier): $\operatorname{curl} v = 0 \Longrightarrow v = \nabla\varphi$. Recall that a vector field v on $\mathcal{U}$ is **conservative** means that for all $a, b \in \mathcal{U}$, the line integral

$$\int_C v \cdot d\vec{r},$$

where $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, is independent of the simple (does not cross itself) curve of integration C from a to b where C lies entirely within $\mathcal{U}$.[16] One can verify that

[15] The formula for β such that $\alpha = d\beta$ above didn't appear out of nothing: We just carefully studied the proof of Theorem 2.50 of Section 2.9 with the homotopy $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined by $H(x,t) = t\,x$, which contracts all of $\mathbb{R}^n$ to the point 0, and from the proof of Theorem 2.50 we derived the formula for β above.

[16] Just so we can use Stokes' theorem from elementary calculus without thinking too much, let us take a "curve" to mean a piecewise infinitely differentiable curve. We can weaken this quite a bit without changing the results.

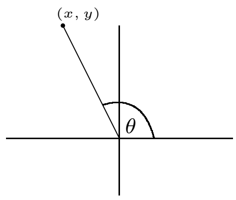

FIGURE 2.17. An angle function.

conservative is equivalent to $\int_C v \cdot d\vec{r} = 0$ for all simple closed curves C in $\mathcal{U}$. Then the true elementary calculus theorem is the following:

Elementary calculus theorem: *A vector field v on $\mathcal{U}$ is conservative if and only if $v = \nabla\varphi$ for some $\varphi \in C^\infty(\mathcal{U}, \mathbb{R})$.*

By Stokes' theorem (which, by the way, we are going to generalize dramatically in Section 2.11) if $\mathcal{U}$ happens to be simply connected, then a vector field v on $\mathcal{U}$ is conservative if and only if $\operatorname{curl} v = 0$. Therefore, for a simply connected region, we have $\operatorname{curl} v = 0 \Longleftrightarrow v = \nabla\varphi$ for some $\varphi \in C^\infty(\mathcal{U}, \mathbb{R})$. Now what if $\mathcal{U}$ is not simply connected? Here is a standard example on what can happen in this situation.

Example 2.38. Consider the punctured plane $\mathbb{P} := \mathbb{R}^2 \setminus \{0\}$ and the vector field

$$\vec{a} = -\frac{y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}. \tag{2.87}$$

Since we are in two dimensions, the curl of a vector field $P\,\vec{i} + Q\,\vec{j}$ is the vector field $(\partial_x Q - \partial_y P)\,\vec{k}$. For the case at hand, we have

$$\partial_x Q = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

and

$$\partial_y P = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2},$$

from which one can check that $\partial_x Q - \partial_y P = 0$. However, consider the line integral of $\vec{a}$ around the loop $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j}$ for $0 \le t \le 2\pi$. On this curve, we have

$$\vec{a} = -\frac{y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j} = -\sin t\,\vec{i} + \cos t\,\vec{j}.$$

Therefore, since $d\vec{r} = -\sin t\,\vec{i} + \cos t\,\vec{j}$, we see that

$$\int_C \vec{a} \cdot d\vec{r} = \int_0^{2\pi} \Big((\sin t)^2 + (\cos t)^2\Big)\,dt = 2\pi.$$

Since this integral is not zero, the vector field $\vec{a}$ is not conservative. In particular, $\vec{a} \neq \nabla\varphi$ for some $\varphi \in C^\infty(\mathcal{U}, \mathbb{R})$. It might be illuminating to remark that $\vec{a}$ is *almost* the gradient of a function, the angle function. For $(x, y) \neq (0, 0)$ off the positive real axis, let $\theta(x, y) =$ angle of the point (x, y) measured from the positive real axis; see Figure 2.17. For example, if $y > 0$, then

$$\theta(x, y) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

It is easy to check that $\vec{a} = \nabla\theta$. However, this does not contradict the previous discussion since θ is not a smooth function on all of $\mathbb{P}$ but only on the subset $\mathbb{R}^2 \setminus ([0,\infty) \times \{0\})$.

Summarizing the above example, we have found a vector field $\vec{a}$ on the punctured plane $\mathbb{P}$ whose curl is zero but is not the gradient of a function. We can interpret this example in the following way. Define the vector spaces

$$Z = \{v \in C^\infty(\mathbb{P}, T\mathbb{P}) \,|\, \text{curl } v = 0\} \ , \quad B = \{\nabla\varphi \,|\, \varphi \in C^\infty(\mathbb{P}, \mathbb{R})\} = \nabla C^\infty(\mathbb{P}, \mathbb{R}).$$

Then we know that

$$B \subseteq Z \ \text{ but there are topological obstructions to } B = Z.$$

Thus, in an intuitive sense, the elements of the vector space Z that are not in the subspace B, which mathematically we can think of as the quotient

$$Z/B,$$

should be a good measurement of the topological obstructions. In the case we are looking at, $\mathbb{P}$, the topological obstruction is the hole at $\{0\}$ so Z/B should be one-dimensional if indeed Z/B does measure the non-trivial topological aspects of $\mathbb{P}$. This is in fact the case and here is a proof, which solves a similar problem written on the blackboard in the movie "A beautiful mind."[17]

THEOREM 2.45. *The vector space Z/B is one-dimensional, spanned by the vector $\vec{a}$ in* (2.87) *of Example 2.38.*

PROOF. This theorem is really part of elementary calculus so we shall leave the proof to you with some hints.

Step 1: First prove the following lemma: If $v \in C^\infty(\mathbb{P}, T\mathbb{P})$ and $\text{curl}\, v = 0$, then for any positively oriented (counter-clockwise) simple closed curves C_1, C_2 surrounding the origin, we have

$$\int_{C_1} v \cdot d\vec{r} = \int_{C_2} v \cdot d\vec{r}.$$

Suggestion: Let C be any large circle around the origin that completely contains C_1 and C_2. Using Stokes' theorem prove that $\int_C v \cdot d\vec{r} = \int_{C_1} v \cdot d\vec{r}$ and that $\int_C v \cdot d\vec{r} = \int_{C_2} v \cdot d\vec{r}$.

Step 2: For any $v \in C^\infty(\mathbb{P}, T\mathbb{P})$ with $\text{curl}\, v = 0$, define

$$w := v - \frac{c}{2\pi}\,\vec{a},$$

where $\vec{a}$ is the vector in (2.87) and where $c = \int_C v \cdot d\vec{r}$ where C is any positively oriented simple closed curve around the origin; by *Step 1*, the number c is defined independent of the choice of such a C. Prove that w is conservative. (Hint: Note that $\int_C \vec{a} \cdot d\vec{r} = 2\pi$ for any positively oriented simple closed curve around the origin.) Hence $w = \nabla\varphi$ for some $\varphi \in C^\infty(\mathbb{P}, \mathbb{R})$. □

[17]See the web site `http://www.haverford.edu/math/lbutler/MITclassroom.html` for a video clip of the MIT blackboard scene. Nash seems to be doing a three-dimensional version of the theorem on a blackboard although the notation in the movie is not clear (e.g. what is "X" on the blackboard?).

The vector space $H^1(\mathbb{P}) := Z/B$ is called the first de Rham cohomology space of $\mathbb{P}$. Thus, we have proved that $H^1(\mathbb{P})$ is one-dimensional, which reflects the "hole" at the origin. By the "Elementary calculus theorem" we know that $H^1(\mathbb{R}^2) = 0$ (in fact, $H^1(\mathcal{U}) = 0$ for any simply connected domain $\mathcal{U} \subseteq \mathbb{R}^2$), which reflects the trivial topology of $\mathbb{R}^2$.

2.8.2. The de Rham cohomology. For a general manifold, grad, curl, and div are contained in the exterior derivative so we can easily generalize the above considerations to any manifold. For a manifold M and a nonnegative integer k we define the vector spaces

$$\boxed{\begin{aligned} Z^k(M) &:= \{\alpha \in C^\infty(M, \Lambda^k) \mid d\alpha = 0\} \\ &= \ker\Big(d : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k+1})\Big) \end{aligned}}$$

and

$$\boxed{\begin{aligned} B^k(M) &:= dC^\infty(M, \Lambda^{k-1}) \\ &= \operatorname{Im}\Big(d : C^\infty(M, \Lambda^{k-1}) \to C^\infty(M, \Lambda^k)\Big); \end{aligned}}$$

if $k = 0$ we put $B^0(M) := 0$. Here, Z stands for the German word "Zyklus" meaning "cycle" and B stands for "boundary" since the elements of Z^k are called "co-cycles" and the elements of B^k are called "co-boundaries" for general cohomology theories studied in algebraic topology. We also call elements of $Z^k(M)$ **closed forms** and elements of $B^k(M)$ **exact forms**. Observe that $B^k(M) \subseteq Z^k(M)$ since $d(d\beta) = d^2\beta = 0$ for any $\beta \in C^\infty(M, \Lambda^{k-1})$. The difference between $B^k(M)$ and $Z^k(M)$ should measure, just as in the above discussion with the punctured plane, the nontrivial topological aspects of M. This leads us to consider the quotient space

$$\boxed{H^k(M) := \frac{Z^k(M)}{B^k(M)} = \frac{\{\alpha \in C^\infty(M, \Lambda^k) \mid d\alpha = 0\}}{dC^\infty(M, \Lambda^{k-1})}.}$$

If $\alpha \in C^\infty(M, \Lambda^k)$ is closed, the coset

$$[\alpha] = \alpha + dC^\infty(M, \Lambda^{k-1})$$

is called the **cohomology class** of α. The space $H^k(M)$ is called the **de Rham cohomology space** (of degree k) named after Georges de Rham (1903–1990). Note that $H^k(M)$ is a real vector space, being the quotient of two real vector spaces $B^k(M) \subseteq Z^k(M)$. Note that when $k = n$ where $n = \dim M$, then all n forms are exact (because there are no $(n+1)$-forms), hence

$$H^n(M) = \frac{C^\infty(M, \Lambda^k)}{dC^\infty(M, \Lambda^{k-1}M)} , \quad n = \dim M.$$

Finally, we put

$$\boxed{H^*(M) := H^0(M) \oplus H^1(M) \oplus \cdots \oplus H^n(M).}$$

If $\mathcal{U} \subseteq \mathbb{R}^3$ is open, then de Rham spaces can be stated as

$$H^1(\mathcal{U}) = \frac{\{v \in C^\infty(\mathcal{U}, T\mathcal{U}) \mid \operatorname{curl} v = 0\}}{\nabla C^\infty(\mathcal{U}, \mathbb{R})} \quad , \quad H^2(\mathcal{U}) = \frac{\{v \in C^\infty(\mathcal{U}, T\mathcal{U}) \mid \operatorname{div} v = 0\}}{\operatorname{curl} C^\infty(\mathcal{U}, T\mathcal{U})}.$$

Although it is not easy to compute the de Rham cohomology spaces for general manifolds, some of these spaces are easy to compute explicitly.

Proposition 2.46. *For any manifold M, $H^0(M)$ is exactly the space of functions constant on the connected components of M. In particular, if M has m connected components, then $H^0(M) \cong \mathbb{R}^m$. If $k > \dim M$, then $H^k(M) = 0$.*

Proof. Since $\Lambda^k M = 0$ for $k > \dim M$, the last statement is immediate from the definition. For $k = 0$, since $B^0(M) := 0$, we have

$$H^0(M) := Z^0(M)/B^0(M) = \{f \in C^\infty(M, \mathbb{R}) \mid df = 0\}.$$

If $(\mathcal{U}, x)$ is a coordinate patch, then $df = \sum_{j=1}^n \partial_{x_j} f \, dx_j$ and hence $df = 0$ if and only if $\partial_{x_j} f = 0$ for all j, if and only if f is constant on the connected components of $\mathcal{U}$. Using this fact, it follows that $df = 0$ if and only if f is constant on the connected components of M. □

As stated in this proposition, $H^0(M)$ measures the number of connected components of M. We intuitively think of the higher degree cohomology spaces as follows. We think of $H^1(M)$ as measuring the number of "holes" in M, which measures the obstructions to a circle being able to shrink to a point in M. We think of $H^2(M)$ as measuring the number of "hollow cavities," which measures the obstructions to a sphere being able to shrink to a point in M. $H^k(M)$ for $k > 2$ measures similar higher-dimensional analogs of "holes".

Example 2.39. By our discussion in the previous subsection, we know that $H^1(\mathbb{R}^2) = 0$ while $H^1(\mathbb{P}) \cong \mathbb{R}$, which measures the fact that a circle around the origin cannot be shrunk to a point without leaving the punctured plane. By our proposition we have $H^0(\mathbb{R}^2) = \mathbb{R}$ and $H^0(\mathbb{P}) = \mathbb{R}$. Later on (see Examples 2.44 and 2.45) we shall see that $H^2(\mathbb{R}^2) = H^2(\mathbb{P}) = 0$, which simply says that $\mathbb{R}^2$ and $\mathbb{P}$) have no hollow cavities. By our proposition, $H^k(\mathbb{R}^2) = H^k(\mathbb{P}) = 0$ for all $k \geq 3$.

On the other hand, in Subsection 2.9.3 we shall see that $H^2(\mathbb{S}^2) \cong \mathbb{R}$, which says that $\mathbb{S}^2$ does have a hollow cavity as we already know!

Example 2.40. Consider now the circle. Since $\mathbb{S}^1$ is connected and one-dimensional we know that $H^0(\mathbb{S}^1) = \mathbb{R}$ and $H^k(\mathbb{S}^1) = 0$ for $k \geq 2$. Thus, we just have to compute $H^1(\mathbb{S}^1)$, which should equal $\mathbb{R}$ since $\mathbb{S}^1$ has a hole. We now compute $H^1(\mathbb{S}^1)$ using a technique familiar to you if you did Problem 1 in Exercises 1.1. We first define a one-form $d\theta$ on $\mathbb{S}^1$. There are many ways to define $d\theta$. One way to define $d\theta$ is as the dual element of vector field ∂_θ found in Problem 2 of Exercises 2.4. We can also define $d\theta$ using angular coordinates that we introduced in Example 2.14 in Section 2.3. To this end, let $a \in \mathbb{R}$ and consider the coordinate patch

$$F_a : \mathcal{U}_a \to \mathcal{V}_a, \quad \text{where } \mathcal{U}_a = \mathbb{S}^1 \setminus \{(\cos a, \sin a)\}, \quad \mathcal{V} = (a, a + 2\pi), \tag{2.88}$$

defined by $F_a(p) := \theta_a$ where θ_a is the unique real number in the interval $(a, a+2\pi)$ such that $p = (\cos\theta_a, \sin\theta_a)$. In other words, we just assign to p its angle in the interval $(a, a + 2\pi)$ measured from the positive real axis. (In Example 2.14 we looked at the particular case $a = 0$.) Because the angles of any given point differ by integer multiples of 2π, it follows that for any $a, b \in \mathbb{R}$, for some $k \in \mathbb{Z}$ we have $\theta_a(p) = \theta_b(p) + 2\pi k$ for all $p \in \mathcal{U}_a \cap \mathcal{U}_b$. In particular, $d\theta_a = d\theta_b$ on $\mathcal{U}_a \cap \mathcal{U}_b$. Hence, we can define a one-form $d\theta$ on $\mathbb{S}^1$ as follows: If $p \in \mathbb{S}^1$ choose any $a \in \mathbb{R}$ such that $p \in \mathcal{U}_a$, then define $d\theta(p) := d\theta_a(p)$; this is well-defined independent of the choice of a. We now compute

$$H^1(\mathbb{S}^1) = \frac{C^\infty(\mathbb{S}^1, \Lambda^1)}{dC^\infty(\mathbb{S}^1, \mathbb{R})}.$$

If $\alpha \in C^\infty(\mathbb{S}^1, \Lambda^1\mathbb{S}^1)$, then we can write $\alpha = f\, d\theta$ where $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$. In particular, since $\mathbb{R} \ni t \mapsto (\cos t, \sin t)$ is smooth (this can be readily checked) the composition $\mathbb{R} \ni t \mapsto f(\cos t, \sin t)$ is a smooth function of $t \in \mathbb{R}$. We define a map

$$I : C^\infty(\mathbb{S}^1, \Lambda^1) \to \mathbb{R} \quad \text{by} \quad I(\alpha) := \int_0^{2\pi} f(\cos t, \sin t)\, dt.$$

If $\alpha = dg$ is exact, then using the coordinate patch F_a in (2.88) with $a = 0$ we have $\alpha = \partial_\theta g\, d\theta$ where $\partial_\theta g := \partial_\theta(g \circ F^{-1}) = \partial_\theta\big(g(\cos\theta, \sin\theta)\big)$. Hence,

$$\begin{aligned} I(\alpha) = \int_0^{2\pi} \partial_t\big(g(\cos t, \sin t)\big)\, dt &= g(\cos 2\pi, \sin 2\pi) - g(\cos 0, \sin 0) \\ &= g(1,0) - g(1,0) = 0. \end{aligned}$$

Therefore, I descends to a map on the quotient

$$I : H^1(\mathbb{S}^1) = \frac{C^\infty(\mathbb{S}^1, \Lambda^1)}{dC^\infty(\mathbb{S}^1, \mathbb{R})} \to \mathbb{R}.$$

Notice that I is surjective because for any constant $c \in \mathbb{R}$, we have

$$I(c\, d\theta) = \int_0^{2\pi} c\, dt = 2\pi c.$$

We claim that I is injective, which shows that $H^1(\mathbb{S}^1) \cong \mathbb{R}$. So, let $[\alpha] \in H^1(\mathbb{S}^1)$ where $\alpha = f\, d\theta$ and suppose that $\int_0^{2\pi} f(\cos t, \sin t)\, dt = 0$. Define $g : \mathbb{S}^1 \to \mathbb{R}$ as follows: If $p \in \mathbb{S}^1$ write $p = (\cos\theta, \sin\theta)$ for $\theta \in \mathbb{R}$ and define

$$g(p) := \int_0^\theta f(\cos t, \sin t)\, dt.$$

Of course as it stands, $g(p)$ may not be well defined because for any $k \in \mathbb{Z}$, we have $p = (\cos(\theta + 2\pi k), \sin(\theta + 2\pi k))$. However, since $\int_0^{2\pi} f(\cos t, \sin t)\, dt = 0$, we have

$$\begin{aligned} \int_0^{\theta+2\pi k} f(\cos t, \sin t)\, dt &= \int_0^{2\pi k} f(\cos t, \sin t)\, dt + \int_{2\pi k}^{\theta+2\pi k} f(\cos t, \sin t)\, dt \\ &= 0 + \int_0^\theta f(\cos(s - 2\pi k), \sin(s - 2\pi k))\, ds, \quad (s = t - 2\pi k) \\ &= \int_0^\theta f(\cos t, \sin t)\, dt. \end{aligned}$$

Thus, $g(p)$ is well-defined. We leave it for you to check that $g : \mathbb{S}^1 \to \mathbb{R}$ is smooth and that $\alpha = dg$. Therefore $[\alpha] = 0$ and hence I is injective.

In conclusion, we know all the cohomology spaces of $\mathbb{S}^1$:

$$H^k(\mathbb{S}^1) = \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & k > 1. \end{cases}$$

2.8.3. Push-forwards and what makes good invariants for manifolds? In order that the de Rham cohomology spaces fit our picture of measuring the non-trivial topological aspects of manifolds, we would like the de Rham spaces to have certain properties. Here are some of the many desirable properties.

(i) The "trivial" manifolds like $\mathbb{R}^n$ should have "trivial" cohomology spaces. This is reflected in Poincaré's lemma as seen in Theorem 2.53 and Example 2.45. Similarly, non-trivial manifolds should have non-trivial cohomologies. We already saw this for the de Rham spaces of $\mathbb{S}^1$ and $\mathbb{P}$.

(ii) Diffeomorphic manifolds should have the same cohomologies. Thus, if M and N are diffeomorphic manifolds, then $H^k(M)$ and $H^k(N)$ should be isomorphic. This is the case as seen in Theorem 2.49 in Section 2.8.5. In this sense, we can consider the de Rham cohomology spaces as "invariants" because they are independent of the manifold up to diffeomorphism. Even better ...

(iii) Instead of diffeomorphic, it would be even nicer that homotopy equivalent manifolds should have the same cohomologies. Thus, if M and N are homotopy equivalent manifolds, then $H^k(M)$ and $H^k(N)$ should be isomorphic. This is the case as seen in Section 2.9.

(iv) Smooth maps between manifolds should induce maps between cohomology spaces. That is, if M and N are manifolds, then smooth maps between M and N should induce linear maps between $H^k(M)$ and $H^k(N)$; this allows the spaces $H^k(M)$ to be studied from the spaces $H^k(N)$ or vise versa. This is topic of Section 2.8.5.

(v) One should be able to understand the cohomology of a manifold by knowing the cohomology on pieces of M. More precisely, if a manifold M is broken up into smaller pieces, let us say that $M = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}, \mathcal{V} \subseteq M$ are open, then we should be able to understand the cohomology spaces of M from those of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{U} \cap \mathcal{V}$. This is true in principle via the Mayer-Vietoris sequence to be studied in Section 2.9.

The goal of this subsection is to formalize Topic (iv). Let $f : M \to N$ be a smooth map between manifolds. We begin by defining a map $f_* : TM \to TN$ as a map such that for each $p \in M$,

$$f_* : T_pM \to T_{f(p)}N.$$

If we want to emphasize the base point we can denote this map by $(f_*)_p$. To define this map, fix $p \in M$ and let $v \in T_pM$; we shall define $f_*v \in T_{f(p)}N$. Recall that for any manifold X and point $q \in X$, T_qX consists of linear maps $w : C_q(X) \to \mathbb{R}$ satisfying the product rule, where

$$C_q(X) := \{\varphi : \mathcal{U} \to \mathbb{R} \text{ smooth where } \mathcal{U} \subseteq X \text{ is open and } q \in \mathcal{U}\}.$$

Thus, we need to define $(f_*v)(\varphi)$ where $\varphi \in C_{f(p)}(N)$. Notice that the **pullback** of φ under f,

$$\boxed{f^*\varphi := \varphi \circ f,}$$

is a smooth function defined near p (because φ is smooth near $f(p)$); see the top picture in Figure 2.18. That is,

$$f^* : C_{f(p)}(N) \to C_p(M).$$

In particular, $v(f^*\varphi)$ makes sense because $v : C_p(M) \to \mathbb{R}$. We define

$$\boxed{(f_*v)(\varphi) := v(f^*\varphi) \quad \text{for all} \quad \varphi \in C_{f(p)}(N).}$$

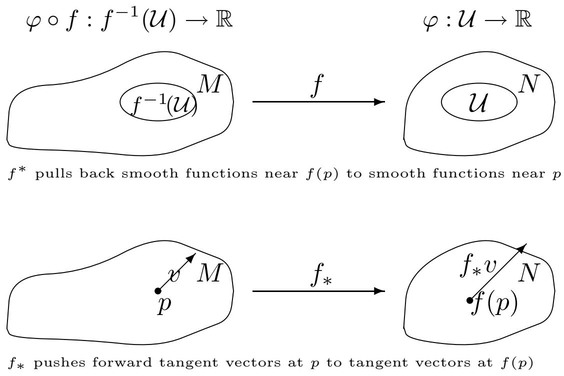

FIGURE 2.18. Illustrating the concepts of pullback of a function φ and pushforward of a vector v.

It's straightforward to check that $f_* v$ is linear and satisfies the product rule, so $f_* v \in T_{f(p)}N$. This defines a map $f_* : T_pM \to T_{f(p)}N$, which you can check to be a linear map between the vector spaces T_pM and $T_{f(p)}N$. Hence, as $p \in M$ was arbitrary, we get a map

$$f_* : TM \to TN,$$

called the **pushforward** (also called the **differential**) of f because it "pushes" vectors from M "forward" to vectors on N; see the bottom picture in Figure 2.18.

Example 2.41. Consider the special case when $N = \mathbb{R}$. Then for any $p \in M$ and $v \in T_pM$, we have $f_*(v) \in T_{f(p)}\mathbb{R}$. If x denotes the standard coordinate on $\mathbb{R}$ (that is, $x(r) = r$ for all $r \in \mathbb{R}$), then ∂_x is a basis for $T_{f(p)}\mathbb{R}$ so we can write

$$f_* v = a\, \partial_x$$

for some $a \in \mathbb{R}$. To find a, we apply both sides to the function $\varphi(x) = x$ and find

$$a = (f_* v)(x) = v(f^* x) = v(x \circ f) = v(f) = df(v),$$

where we used that $x \circ f = f$ since $x(f(q)) = f(q)$ for all $q \in M$. Thus,

$$f_* v = df(v)\, \partial_x.$$

Dropping ∂_x, so in effect identifying $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$, we see that $f_* = df$ at least when $N = \mathbb{R}$. For this reason, some authors denote f_* by df in general, but we shall reserve "d" for the special meaning of the differential of a function or the exterior derivative of a form.

The following example shows that when M and N are just Euclidean space, with respect to the standard coordinate vector fields, f_* is what you probably called Df or Jf, the Jacobian matrix, in undergraduate real analysis.

Example 2.42. Consider a function $f : \mathbb{R}^m \to \mathbb{R}^n$ (so in this case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$), denote by $x = (x_1, \ldots, x_m)$ the coordinate function on $\mathbb{R}^m$ and $y = (y_1, \ldots, y_n)$ the coordinate function on $\mathbb{R}^n$, and write

$$y = f(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

For a point $p \in \mathbb{R}^m$, $\{\partial_{x_j}\}$ forms a basis for $T_p\mathbb{R}^m$, so we focus on a basis vector ∂_{x_j} and finding $f_*(\partial_{x_j}) \in T_{f(p)}\mathbb{R}^n$. Let $\varphi \in C_{f(p)}(\mathbb{R}^n)$ be arbitrary and observe that

$$\begin{aligned}(f_*)_p(\partial_{x_j})(\varphi) &:= (\partial_{x_j})(\varphi \circ f)\\ &= \frac{\partial}{\partial x_j}\bigg|_{x=p} (\varphi(f_1(x), \ldots, f_n(x)),\end{aligned}$$

where we write φ as $\varphi(y_1, \ldots, y_n)$. Applying the chain rule, we obtain

$$(f_*)_p(\partial_{x_j})(\varphi) = \sum_{i=1}^{n} \frac{\partial \varphi(y)}{\partial y_i}\bigg|_{f(p)} \frac{\partial f_i}{\partial x_j}(p).$$

Therefore,

$$(f_*)_p(\partial_{x_j}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p) \left(\frac{\partial}{\partial y_i}\right)_{f(p)},$$

and hence the matrix of $(f_*)_p$ with respect to the coordinate vector fields is

$$Jf := \left[\frac{\partial f_i}{\partial x_j}(p)\right],$$

which is the familiar Jacobian matrix of f.

Here are some elementary properties of the pushforward.

THEOREM 2.47 (**Pushforward properties**). *The pushforward has the following properties:*

(i) *If* $\mathrm{Id} : M \to M$ *is the identity map, then* $(\mathrm{Id}_*)_p = \mathrm{Id} : T_pM \to T_pM$ *for all* $p \in M$.

(ii) *We have* $(f \circ g)_* = f_* \circ g_*$ *for any composition of smooth maps*

$$X \xrightarrow{g} M \xrightarrow{f} N.$$

That is, for any point $p \in X$, $((f \circ g)_*)_p = (f_*)_{g(p)} \circ (g_*)_p$, *or as a diagram:*

$$T_pX \xrightarrow{(g_*)_p} T_{g(p)}M \xrightarrow{(f_*)_{g(p)}} T_{f(g(p))}N$$

$$T_pX \xrightarrow{((f\circ g)_*)_p} T_{f(g(p))}N$$

(iii) *If* $f : M \to N$ *is a diffeomorphism, then at each* $p \in M$, $f_* : T_pM \to T_{f(p)}N$ *is an isomorphism whose inverse is given by* $(f_*)^{-1} = (f^{-1})_*$.

(iv) *Let* $f : M \to N$ *be smooth, where* $m = \dim M$ *and* $n = \dim N$, *and let* $(\mathcal{U}, x)$ *be a coordinate patch containing a point* $p \in M$ *and* $(\widetilde{\mathcal{U}}, y)$ *be a coordinate patch containing* $f(p) \in N$. *Let* $f_i := f^*y_i = y_i \circ f$ *for* $i = 1, \ldots, n$ *be the coordinate functions of* f. *Then* $(f_*)_p(\partial_{x_j}) \in T_{f(p)}N$ *is given by*

$$(f_*)_p(\partial_{x_j}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p)\, \partial_{y_i}\,, \quad j = 1, 2, \ldots, m.$$

In other words, the matrix of $(f_*)_p$ *is*

$$Jf := \left[\frac{\partial f_i}{\partial x_j}(p)\right],$$

which is called the ***Jacobian matrix*** *of* f *with respect to the coordinates* $(\mathcal{U}, x)$ *of* M *and* $(\widetilde{\mathcal{U}}, y)$ *of* N.

PROOF. We shall prove properties *(ii)* and *(iv)* leaving the others for your enjoyment. (Hint: *(iii)* can be proved by combining *(i)* and *(ii)*.) To prove *(ii)*, let $v \in T_pX$. Then for any $\varphi \in C_{f(g(p))}(N)$, we have

$$\begin{aligned} ((f \circ g)_*)_p v(\varphi) := v((f \circ g)^* \varphi) &= v(\varphi \circ f \circ g) \\ &= v(g^*(\varphi \circ f)) \\ &=: ((g_*)_p v)(f^* \varphi) \\ &=: (f_*)_{g(p)}((g_*)_p v)(\varphi), \end{aligned}$$

which proves *(ii)*.

To prove *(iv)*, we just repeat what we did in Example 2.42 just being more careful with notation! Let F be the coordinate patch $(\mathcal{U}, x)$, let G be the coordinate patch $(\widetilde{\mathcal{U}}, y)$ (see e.g. Figure 2.12 of Section 2.2), and write

$$G \circ f = (f_1, f_2, \ldots, f_n),$$

where $f_i = y_i \circ f$. Now let $\varphi \in C_{f(p)}(N)$ be arbitrary and observe that

$$\begin{aligned} (f_*)_p(\partial_{x_j})(\varphi) &:= (\partial_{x_j})(\varphi \circ f) \\ &:= \frac{\partial}{\partial x_j}\bigg|_{x=F(p)} (\varphi \circ f \circ F^{-1}(x) \\ &= \frac{\partial}{\partial x_j}\bigg|_{x=F(p)} (\varphi \circ G^{-1} \circ G \circ f \circ F^{-1}(x)) \\ &= \frac{\partial}{\partial x_j}\bigg|_{x=F(p)} (\varphi(f_1(x), \ldots, f_n(x)), \end{aligned}$$

where we write $\varphi(y_1, \ldots, y_n)$ for $\varphi \circ G^{-1}(y)$ and $f_i(x)$ for $f_i \circ F^{-1}(x)$. Applying the chain rule, we obtain

$$(f_*)_p(\partial_{x_j})(\varphi) = \sum_{i=1}^{n} \frac{\partial \varphi \circ G^{-1}(y)}{\partial y_i}\bigg|_{G(f(p))} \frac{\partial f_i \circ F^{-1}(x)}{\partial x_j}\bigg|_{F(p)}.$$

In usual notation, this is nothing more than

$$(f_*)_p(\partial_{x_j}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p) \left(\frac{\partial}{\partial y_i}\right)_{f(p)},$$

which is exactly what we wanted to show. $\square$

If you like commutative diagrams, the last property simply means that the following diagram commutes:

$$\begin{array}{ccc} T_pM & \xrightarrow{f_*} & T_{f(p)}N \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R}^m & \xrightarrow{Jf} & \mathbb{R}^n \end{array},$$

where the vertical isomorphisms are just the identifications of the tangent spaces with Euclidean space via the bases $\{\partial_{x_j}\}$ and $\{\partial_{y_i}\}$. The exercises contain some problems that use this theorem and that will help you to brush up on the inverse function theorem from advanced calculus.

2.8.4. Pullback of forms. By our (at this point) usual duality trick, since we can pushforward vectors we can then pullback tensors. For a smooth map $f : M \to N$, we know that for any point $p \in M$,

$$f_* : T_pM \to T_{f(p)}N.$$

Given by $\alpha \in T^*_{f(p)}N^{\otimes k}$, which means that $\alpha : \underbrace{T_{f(p)}N \times \cdots \times T_{f(p)}N}_{k \text{ tangent spaces}} \to \mathbb{R}$ is multi-linear, we define

$$f^*\alpha \in T^*_pM^{\otimes k} \quad \Longleftrightarrow \quad f^*\alpha : T_pM \times \cdots \times T_pM \to \mathbb{R} \ \text{ is multi-linear,}$$

by the formula

$$\boxed{(f^*\alpha)(v_1, \ldots, v_k) := \alpha(f_*v_1, \ldots, f_*v_k) \quad \text{for all } v_1, \ldots, v_k \in T_pM.} \tag{2.89}$$

Note that $f_*v_i \in T_{f(p)}N$ for all i, so that $\alpha(f_*v_1, \ldots, f_*v_k)$ is defined. This gives a map

$$f^* : T^*_{f(p)}N^{\otimes k} \to T^*_pM^{\otimes k}.$$

The definition (2.89) preserves the alternating property so

$$f^* : \Lambda^k T^*_{f(p)}N \to \Lambda^k T^*_pM,$$

which is easily checked to be a linear map. If we want to emphasize the base point we can denote this map by f^*_p. The pullback has some very nice and easy to prove properties built into its definition; let's focus on alternating tensors for concreteness. For example, if $\alpha_1, \ldots, \alpha_k \in \Lambda^k T^*_{f(p)}N$, then we claim that

$$\boxed{f^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = (f^*\alpha_1) \wedge \cdots \wedge (f^*\alpha_k).} \tag{2.90}$$

To prove this is easy: If $v_1, \ldots, v_k \in T_pM$, then using the handy formula (2.73) back in Section 2.6.4, we see that

$$\begin{aligned} f^*(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) &:= (\alpha_1 \wedge \cdots \wedge \alpha_k)(f_*v_1, \ldots, f_*v_k) \\ &= \det[\alpha_i(f_*v_j)] \\ &= \det[(f^*\alpha_i)(v_j)] \\ &= (f^*\alpha_1) \wedge \cdots \wedge (f^*\alpha_k). \end{aligned}$$

We can also prove this using the definition of wedge (cf. Proposition 2.31):

$$\begin{aligned} f^*(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) &:= (\alpha_1 \wedge \cdots \wedge \alpha_k)(f_*v_1, \ldots, f_*v_k) \\ &= \sum_\sigma \operatorname{sgn}\sigma\, \alpha_{\sigma(1)}(f_*v_1) \cdots \alpha_{\sigma(n)}(f_*v_k) \\ &= \sum_\sigma \operatorname{sgn}\sigma\, (f^*\alpha_{\sigma(1)})(v_1) \cdots (f^*\alpha_{\sigma(n)})(v_k) \\ &= (f^*\alpha_1) \wedge \cdots \wedge (f^*\alpha_k). \end{aligned}$$

Any case, another important property is that if $g \in C^\infty(N)$, then

$$\boxed{f^*(dg) = d(f^*g),} \tag{2.91}$$

where $f^*g := g \circ f$. To prove this we just use definitions: For any $v \in T_pM$, we have

$$f^*(dg)(v) := dg(f_*v) := (f_*v)(g) := v(f^*g) =: d(f^*g)(v).$$

So easy! We can even pullback smooth tensors. Let's focus on alternating tensors. If $\alpha \in C^\infty(N, \Lambda^k)$, then we define $f^*\alpha \in C^\infty(M, \Lambda^k)$ by

$$\boxed{(f^*\alpha)_p := f_p^* \alpha_{f(p)} \quad \text{for all } p \in M.}$$

Note that $\alpha_{f(p)} \in \Lambda^k T^*_{f(p)} N$ so $f_p^* \alpha_{f(p)} \in \Lambda^k T_p^* M$. We still need to verify that $f^*\alpha$ is indeed smooth. To do so, let us choose any point $p \in M$, let $(\widetilde{\mathcal{U}}, y)$ be any coordinates in N on a neighborhood of $f(p)$, and write

$$\alpha = \sum_I a_I \, dy_I \quad \text{where} \quad a_I \in C^\infty(\widetilde{\mathcal{U}}, \mathbb{R}). \tag{2.92}$$

Then in view of (2.90) and (2.91), at all points of $f^{-1}(\mathcal{U})$ we have

$$\begin{aligned} f^*\alpha &= \sum_I f^* a_I \, f^*(dy_{i_1} \wedge \cdots \wedge dy_{i_k}) \\ &= \sum_I f^* a_I \, (f^* dy_{i_1}) \wedge \cdots \wedge (f^* dy_{i_k}) \\ &= \sum_I f^* a_I \, d(f^* y_{i_1}) \wedge \cdots \wedge d(f^* y_{i_k}). \end{aligned} \tag{2.93}$$

Since f is smooth, all the functions on the right (namely, $f^* a_I$, $f^* y_{i_1}$, ..., $f^* y_{i_k}$) are smooth functions on $f^{-1}(\mathcal{U})$. Hence $f^*\alpha$ is smooth on $f^{-1}(\mathcal{U})$. Since all such sets $f^{-1}(\mathcal{U})$ cover M, it follows that $f^*\alpha$ is a smooth k-form on M; thus,

$$f^* : C^\infty(N, \Lambda^k) \to C^\infty(M, \Lambda^k).$$

We claim that the commutativity property (2.91) holds at the form level:

$$\boxed{f^*(d\alpha) = d(f^*\alpha) \quad \text{for all } \ \alpha \in C^\infty(N, \Lambda^k).}$$

To prove this, write α in local coordinates as in (2.92), so that $f^*\alpha$ takes the form in (2.93), and compute $d(f^*\alpha)$:

$$\begin{aligned} d(f^*\alpha) &= \sum_I d(f^* a_I) \wedge d(f^* y_{i_1}) \wedge \cdots \wedge d(f^* y_{i_k}) \quad (\text{since } d(df^* y_j) = 0 \text{ for all } j) \\ &= \sum_I f^*(da_I) \wedge (f^* dy_{i_1}) \wedge \cdots \wedge (f^* dy_{i_k}) \quad (\text{by (2.91)}) \\ &= \sum_I f^*(da_I \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_k}) \quad (\text{by (2.90)}) \\ &= f^* \left(\sum_I da_I \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_k} \right) = f^* d\alpha. \end{aligned}$$

Let's summarize our findings in the following theorem.

THEOREM 2.48 (**Pullback properties**). *For $f : M \to N$, pullback under f^* is a linear map*

$$f^* : C^\infty(N, \Lambda^k) \to C^\infty(M, \Lambda^k) \quad \textit{for all } k,$$

that has the following properties:

(i) If $\mathrm{Id} : M \to M$ *is the identity map, then* $\mathrm{Id}^* = \mathrm{Id}$*, the identity map.*
(ii) Pullback preserves wedge: $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$.
(iii) Pullback commutes with d: $f^*(d\alpha) = d(f^*\alpha)$.

(iv) *We have* $(f \circ g)^* = g^* \circ f^*$ *for any composition of smooth maps*

$$X \xrightarrow{g} M \xrightarrow{f} N.$$

(v) *If* $f : M \to N$ *is a diffeomorphism, then the pullback* f^* *is an isomorphism whose inverse is given by* $(f^*)^{-1} = (f^{-1})^*$.

We haven't proved the last two properties, but we shall leave them for your enjoyment; they are really easy: just use Properties *(ii)* and *(iii)* of Theorem 2.47, and *(v)* can also be proved from *(i)* and *(iv)* by applying the pullback operation to $f \circ f^{-1} = \mathrm{Id}$ and $f^{-1} \circ f = \mathrm{Id}$.

Example 2.43. Let $f : M \to N$ be smooth, where $n = \dim M = \dim N$, and let $(\mathcal{U}, x)$ be a coordinate patch containing a point $p \in M$ and $(\widetilde{U}, y)$ be a coordinate patch containing $f(p) \in N$. Let $f_i := f^* y_i = y_i \circ f$ for $i = 1, \ldots, n$ be the coordinate functions of f. Then

$$f^*(dy_i) = d(f^* y_i) = df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, dx_j.$$

In particular, by (2.74) back in Section 2.6.4, we see that

$$\begin{aligned} f^*(dy_1 \wedge \cdots \wedge dy_n) &= \det \left[\frac{\partial f_i}{\partial x_j} \right] dx_1 \wedge \cdots \wedge dx_n \\ &= \det Jf\, dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Another way to prove this is to note that $f^*(dy_1 \wedge \cdots \wedge dy_n) = a\, dx_1 \wedge \cdots \wedge dx_n$ for some function a. To find a, apply both sides to $(\partial_{x_1}, \ldots, \partial_{x_n})$, in which case we get

$$\begin{aligned} a = f^*(dy_1 \wedge \cdots \wedge dy_n)(\partial_{x_1}, \ldots, \partial_{x_n}) &= (dy_1 \wedge \cdots \wedge dy_n)(f_*(\partial_{x_1}), \ldots, f_*(\partial_{x_n})) \\ &= \det \left[\frac{\partial f_i}{\partial x_j} \right], \end{aligned}$$

where we used that $(f_*)(\partial_{x_j}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}\, \partial_{y_i}$ from Theorem 2.47 and the formula for the determinant (2.70) in Section 2.6.4. The formula

$$\boxed{f^*(dy_1 \wedge \cdots \wedge dy_n) = \det Jf\, dx_1 \wedge \cdots \wedge dx_n} \tag{2.94}$$

will be needed in Section 2.11.

Now comes the anticipated ...

2.8.5. Diffeomorphism invariance of the de Rham cohomology. Let $f : M \to N$, a smooth map between manifolds. If $\alpha \in C^\infty(N, \Lambda^k)$ is closed (that is, $d\alpha = 0$), then $f^*\alpha \in C^\infty(M, \Lambda^k)$ is also closed because d and f^* commute:

$$d(f^*\alpha) = f^*(d\alpha) = f^*(0) = 0.$$

Thus,

$$f^* : Z^k(N) \to Z^k(M).$$

Similarly for an exact form $d\beta \in C^\infty(N, \Lambda^k)$, we have $f^*(d\beta) = d(f^*\beta)$, which is also exact. Thus,

$$f^* : B^k(N) \to B^k(M).$$

Therefore, the pullback descends to a map on the quotient:

$$\boxed{f^* : H^k(N) \to H^k(M)\ , \quad H^k(N) \ni [\alpha] \mapsto f^*[\alpha] := [f^*\alpha] \in H^k(M),}$$

which is well-defined independent of the choice of representative α in a class $[\alpha] = \alpha + dC^\infty(M, \Lambda^k)$. We summarize this discussion in the following theorem. (Properties *(ii)* and *(iii)* below follow from Properties *(iv)* and *(v)* of Theorem 2.48.)

THEOREM 2.49 (**Pullback on cohomology**). *For $f : M \to N$, pullback under f^* is a linear map*

$$f^* : H^k(N) \to H^k(M) \quad \textit{for all } k,$$

that has the following properties:

(i) If $\mathrm{Id} : M \to M$ *is the identity, then* $\mathrm{Id}^* = \mathrm{Id}$*, the identity on* $H^k(M)$*.*
(ii) We have $(f \circ g)^* = g^* \circ f^*$ *for any composition of smooth maps*

$$X \xrightarrow{g} M \xrightarrow{f} N.$$

Thus,

$$\begin{array}{ccccc} H^k(N) & \xrightarrow{f^*} & H^k(M) & \xrightarrow{g^*} & H^k(X) \\ & \searrow & \xrightarrow{(f\circ g)^*} & \nearrow & \end{array}$$

(iii) If $f : M \to N$ *is a diffeomorphism, then* $f^* : H^k(N) \to H^k(M)$ *is an isomorphism whose inverse is given by* $(f^*)^{-1} = (f^{-1})^* : H^k(M) \to H^k(N)$*.*
(iv) Thus, diffeomorphic manifolds have isomorphic de Rham cohomology spaces.

Manifolds that are homeomorphic as topological spaces also have isomorphic de Rham cohomologies, but this is another story; see e.g. [**9**, Ch. 5].

EXERCISES 2.8.

1. Using only tools from elementary calculus (e.g. the elementary calculus theorem), compute (be rigorous for (i) but not too rigorous for (ii)–(iv)) the first cohomology spaces of the following subspaces of Euclidean space. One can use the Mayer-Vietoris sequence in the next section to rigorously compute all the cohomology spaces of (ii)–(iv); see Problem 4 in Exercises 2.9.
   (i) $H^1(\mathbb{R}^1)$.
   (ii) Let $a, b \in \mathbb{R}^2$ be distinct and argue that
   $$H^1(\mathbb{R}^2 \setminus \{a, b\}) \cong \mathbb{R}^2.$$
   (iii) $H^1(M)$ where M is the interior of the solid torus. To be concrete, let's take the solid torus to be the set
   $$M = \mathcal{U} \times (-1, 1) \subseteq \mathbb{R}^3 \quad , \quad \text{where} \quad \mathcal{U} = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}.$$
   (iv) Finally, $H^1(M)$ where $M = \{(x, y, z) \in \mathbb{R}^3 \mid 1 < x^2 + y^2 + z^2 < 2\}$, a thickened spherical shell.
2. If $M = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_m$ is a finite union of *disjoint* open subsets of M, prove that
   $$H^k(M) \cong H^k(\mathcal{U}_1) \oplus \cdots \oplus H^k(\mathcal{U}_m), \quad k = 0, 1, 2, \ldots.$$
3. (**Cup product**) Let $\alpha, \beta \in C^\infty(M, \Lambda)$. Prove that if α and β are closed, then $\alpha \wedge \beta$ is also closed. Prove that if either α or β is exact, then $\alpha \wedge \beta$ is also exact. Conclude that the map
   $$H^*(M) \times H^*(M) \ni ([\alpha], [\beta]) \ \mapsto \ [\alpha] \smile [\beta] := [\alpha \wedge \beta] \in H^*(M)$$
   is defined independent of the choice of representatives α and β of the classes $[\alpha], [\beta]$. This map is called the **cup product**. Prove that $H^*(M)$ with the usual addition and with multiplication defined by $\smile$ is a ring. Is it a commutative ring?

4. (**Inverse function theorem**) Let $f : M \to N$ be a smooth map and suppose that at a point $p \in M$,
$$(f_*)_p : T_pM \to T_{f(p)}N$$
is invertible; this implies in particular that $\dim M = \dim N$. This is equivalent to the Jacobian matrix $[\frac{\partial f_i}{\partial x_j}]$ being invertible at p for any choice of coordinates on M and N defined on a neighborhood of p and $f(p)$, respectively. Prove that there are neighborhoods $\mathcal{U} \subseteq M$ and $\mathcal{V} \subseteq N$ with $p \in \mathcal{U}$ and $f(p) \in \mathcal{V}$ such that
$$f : \mathcal{U} \to \mathcal{V}$$
is a diffeomorphism. Suggestion: This is really just an exercise in using the (smooth version of the) inverse function theorem from undergraduate real analysis. Just in case you forgot, here it is: Let $f : \mathcal{D} \to \mathbb{R}^n$ be smooth where $\mathcal{D} \subseteq \mathbb{R}^n$ is open and suppose that the Jacobian matrix $Jf(p)$ is invertible at a point $p \in \mathcal{D}$. Then f is a local diffeomorphism near p in the sense that there is an open set $\mathscr{D} \subseteq \mathcal{D}$ containing p and an open set $\widetilde{\mathscr{D}}$ containing $f(p)$ such that $f : \mathscr{D} \to \widetilde{\mathscr{D}}$ is a diffeomorphism.
5. Here is a related problem. Let M be a manifold, $\mathcal{W} \subseteq M$ be open, and let $f_1, \ldots, f_n : \mathcal{W} \to \mathbb{R}$ be $n = \dim M$ functions having independent differentials a point $p \in \mathcal{W}$ (that is, $df_1, \ldots, df_n \in T_p^*M$ are independent and hence form a basis of T_p^*M). Define
$$F : \mathcal{W} \to \mathbb{R}^n \quad \text{by} \qquad F := (f_1, \ldots, f_n).$$
Prove that there is an open set $\mathcal{U} \subseteq \mathcal{W}$ containing p and an open set $\mathcal{V} \subseteq \mathbb{R}^n$ such that $F : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, which implies that $F : \mathcal{U} \to \mathcal{V}$ is a coordinate patch on M. We can rephrase this fact as follows: If $df_1, \ldots, df_n \in T_p^*M$ are independent, then $f_1, \ldots, f_n$ form the coordinates of a coordinate patch defined on a neighborhood of p.
6. Here is yet another related problem. Let M be a manifold, $\mathcal{W} \subseteq M$ be open, and let $f_1, \ldots, f_k : \mathcal{W} \to \mathbb{R}$ be functions having independent differentials a point $p \in \mathcal{W}$ (in particular, $k \leq \dim M$). Prove that there is a coordinate patch defined on a neighborhood of p having $f_1, \ldots, f_k$ as the first k coordinate functions.

2.9. The de Rham cohomology II: Homotopy and Mayer-Vietoris

In this section we prove that the de Rham spaces are homotopy invariants and we study the Mayer-Vietoris sequence, named after Walther Mayer (1887–1948) and Leopold Vietoris (1891–2002) (yes, this is no mistake, he lived to be 110 years old!). The Mayer-Vietoris sequence is a useful tool that can help compute de Rham spaces for manifolds by breaking up the manifold into easier to handle subsets.

Section objectives: The student will be able to ...

- prove that two manifolds are homotopy equivalent.
- know the "homological algebra free" proof of the Mayer-Vietoris theorem.
- apply the Mayer-Vietoris sequence to find de Rham spaces.

2.9.1. The homotopy operator.

By Theorem 2.49 we know that diffeomorphic manifolds have isomorphic de Rham cohomology spaces. An even deeper fact is that homotopy equivalent manifolds have isomorphic de Rham cohomology spaces. Before defining "homotopy equivalent," which we'll defined in Section 2.9.2 later, let's define homotopic maps. We say that two maps $f, g : M \to N$ are (smoothly) **homotopic**, written $f \simeq g$, if there is a smooth map
$$H : M \times \mathbb{R} \to N$$
such that
$$H(p, 0) = f(p) \;\text{ for all } p \in M \quad \text{and} \quad H(p, 1) = g(p) \;\text{ for all } p \in M.$$