4. (**Inverse function theorem**) Let $f : M \to N$ be a smooth map and suppose that at a point $p \in M$,
$$(f_*)_p : T_pM \to T_{f(p)}N$$
is invertible; this implies in particular that $\dim M = \dim N$. This is equivalent to the Jacobian matrix $[\frac{\partial f_i}{\partial x_j}]$ being invertible at p for any choice of coordinates on M and N defined on a neighborhood of p and $f(p)$, respectively. Prove that there are neighborhoods $\mathcal{U} \subseteq M$ and $\mathcal{V} \subseteq N$ with $p \in \mathcal{U}$ and $f(p) \in \mathcal{V}$ such that
$$f : \mathcal{U} \to \mathcal{V}$$
is a diffeomorphism. Suggestion: This is really just an exercise in using the (smooth version of the) inverse function theorem from undergraduate real analysis. Just in case you forgot, here it is: Let $f : \mathcal{D} \to \mathbb{R}^n$ be smooth where $\mathcal{D} \subseteq \mathbb{R}^n$ is open and suppose that the Jacobian matrix $Jf(p)$ is invertible at a point $p \in \mathcal{D}$. Then f is a local diffeomorphism near p in the sense that there is an open set $\mathscr{D} \subseteq \mathcal{D}$ containing p and an open set $\widetilde{\mathscr{D}}$ containing $f(p)$ such that $f : \mathscr{D} \to \widetilde{\mathscr{D}}$ is a diffeomorphism.
5. Here is a related problem. Let M be a manifold, $\mathcal{W} \subseteq M$ be open, and let $f_1, \ldots, f_n : \mathcal{W} \to \mathbb{R}$ be $n = \dim M$ functions having independent differentials a point $p \in \mathcal{W}$ (that is, $df_1, \ldots, df_n \in T_p^*M$ are independent and hence form a basis of T_p^*M). Define
$$F : \mathcal{W} \to \mathbb{R}^n \quad \text{by} \qquad F := (f_1, \ldots, f_n).$$
Prove that there is an open set $\mathcal{U} \subseteq \mathcal{W}$ containing p and an open set $\mathcal{V} \subseteq \mathbb{R}^n$ such that $F : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, which implies that $F : \mathcal{U} \to \mathcal{V}$ is a coordinate patch on M. We can rephrase this fact as follows: If $df_1, \ldots, df_n \in T_p^*M$ are independent, then $f_1, \ldots, f_n$ form the coordinates of a coordinate patch defined on a neighborhood of p.
6. Here is yet another related problem. Let M be a manifold, $\mathcal{W} \subseteq M$ be open, and let $f_1, \ldots, f_k : \mathcal{W} \to \mathbb{R}$ be functions having independent differentials a point $p \in \mathcal{W}$ (in particular, $k \leq \dim M$). Prove that there is a coordinate patch defined on a neighborhood of p having $f_1, \ldots, f_k$ as the first k coordinate functions.

2.9. The de Rham cohomology II: Homotopy and Mayer-Vietoris

In this section we prove that the de Rham spaces are homotopy invariants and we study the Mayer-Vietoris sequence, named after Walther Mayer (1887–1948) and Leopold Vietoris (1891–2002) (yes, this is no mistake, he lived to be 110 years old!). The Mayer-Vietoris sequence is a useful tool that can help compute de Rham spaces for manifolds by breaking up the manifold into easier to handle subsets.

SECTION OBJECTIVES: THE STUDENT WILL BE ABLE TO ...

- prove that two manifolds are homotopy equivalent.
- know the "homological algebra free" proof of the Mayer-Vietoris theorem.
- apply the Mayer-Vietoris sequence to find de Rham spaces.

2.9.1. The homotopy operator.

By Theorem 2.49 we know that diffeomorphic manifolds have isomorphic de Rham cohomology spaces. An even deeper fact is that homotopy equivalent manifolds have isomorphic de Rham cohomology spaces. Before defining "homotopy equivalent," which we'll defined in Section 2.9.2 later, let's define homotopic maps. We say that two maps $f, g : M \to N$ are (smoothly) **homotopic**, written $f \simeq g$, if there is a smooth map
$$H : M \times \mathbb{R} \to N$$
such that
$$H(p, 0) = f(p) \text{ for all } p \in M \quad \text{and} \quad H(p, 1) = g(p) \text{ for all } p \in M.$$

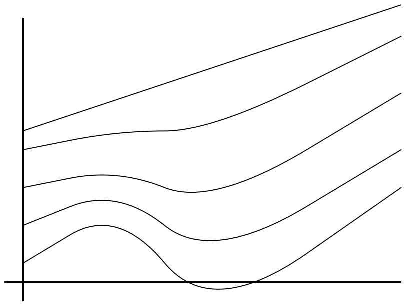


FIGURE 2.19. An example of H_t "deforming" f to g where we show H_t at various $0 < t_1 < t_2 < t_3 < 1$.

Thus, $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. Thinking of t as a parameter and considering the map

$$H_t : M \to N \quad \text{defined by} \quad H_t(p) := H(p,t) \quad \text{for all } p \in M,$$

we have $H_0 = f$ and $H_1 = g$ so we can think of H_t as "smoothly deforming" f to g; see Figure 2.19. We remark that "homotopic" is symmetric: $f \simeq g$ if and only if $g \simeq f$ and even an equivalence relation on the set of smooth maps from M to N. We also remark that the definition of homotopy is usually stated with $\mathbb{R}$ replaced by $[0, 1]$ so that $H : M \times [0, 1] \to N$, but then we have to deal with the boundary points $0, 1$ (and hence we are working with a *manifold with boundary*). To avoid dealing with this topic at the moment we use $\mathbb{R}$ instead of $[0, 1]$. Now, before proving the deep fact concerning homotopy equivalent manifolds and their de Rham cohomology spaces we need to introduce a homotopy operator on differential forms. This requires us to understand differential forms on the manifold $M \times \mathbb{R}$.

In particular, we need the observation that a k-form $\alpha \in C^\infty(M \times \mathbb{R}, \Lambda^k)$ can be written as

$$\alpha = \alpha_0(t) + dt \wedge \alpha_1(t), \tag{2.95}$$

where for each $t \in \mathbb{R}$, $\alpha_0(t) \in C^\infty(M, \Lambda^k)$ and $\alpha_1(t) \in C^\infty(M, \Lambda^{k-1})$.[18] Of course, if $k = 0$, then $\alpha_1(t) \equiv 0$ since there are no -1-forms. To see this, assume for the moment that (2.95) holds and then contract both sides of (2.95) with ∂_t to get

$$\begin{aligned}
\iota_{\partial_t}\alpha &= \iota_{\partial_t}\alpha_0(t) + \iota_{\partial_t}(dt \wedge \alpha_1(t)) \\
&= \iota_{\partial_t}\alpha_0(t) + (\iota_{\partial_t} dt) \wedge \alpha_1(t) - dt \wedge (\iota_{\partial_t}\alpha_1(t)) \\
&= \alpha_1(t),
\end{aligned}$$

where we used the anti-derivation property of contraction (Theorem 2.35) and that $\alpha_0(t)$ and $\alpha_1(t)$ don't have any dt's in them so their contraction with ∂_t is zero. In view of this calculation, we can simply *define*

$$\alpha_1(t) := \iota_{\partial_t}\alpha \quad \text{and} \quad \alpha_0(t) := \alpha - dt \wedge (\iota_{\partial_t}\alpha).$$

[18]Note that $C^\infty(M \times \mathbb{R}, \Lambda^k)$ is short hand for $C^\infty(M \times \mathbb{R}, \Lambda^k(M \times \mathbb{R}))$ and $C^\infty(M, \Lambda^k)$ is shorthand for $C^\infty(M, \Lambda^k M)$; the notation $C^\infty(N, \Lambda^k)$ for any manifold N always means $C^\infty(N, \Lambda^k N) = C^\infty(N, \Lambda^k(T^*N))$.

In local coordinates, $\alpha_0(t)$ and $\alpha_1(t)$ are described as follows. Let $(\mathcal{U}, x)$ be a coordinate patch on M. Then $(\mathcal{U}\times\mathbb{R}, (x,t))$ are coordinates on $M\times\mathbb{R}$, so we know that $dt, dx_1, \ldots, dx_n$ form a basis of $T^*(M\times\mathbb{R})$. Hence, we can write

$$\alpha = \sum_I a_I(x,t)\, dx_I + \sum_J b_J(x,t)\, dt\wedge dx_J \tag{2.96}$$

where the first sum is over all $1 \le i_1 < \cdots < i_k \le n$ and the second sum over $1 \le j_1 < \cdots < j_{k-1} \le n$, and where $a_I(x,t), b_J(x,t) \in C^\infty(\mathcal{U}\times\mathbb{R},\mathbb{R})$. Then

$$\alpha_1(t) := \iota_{\partial_t}\alpha = \sum_J b_J(x,t)\, dx_J$$

and therefore

$$\alpha_0(t) = \alpha - dt\wedge(\iota_{\partial_t}\alpha) = \sum_I a_I(x,t)\, dx_I.$$

Thus, for each $t\in\mathbb{R}$, $\alpha_0(t) \in C^\infty(M,\Lambda^k)$ and $\alpha_1(t)\in C^\infty(M,\Lambda^{k-1})$ exactly as claimed!

We are now ready to define the homotopy operator. We define

$$\boxed{\begin{gathered} Q : C^\infty(M\times\mathbb{R},\Lambda^k) \to C^\infty(M,\Lambda^{k-1}) \\ \text{by}\quad Q\alpha := \int_0^1 \iota_{\partial t}\alpha(t)\, dt, \end{gathered}}$$

where $\iota_{\partial_t}\alpha = \alpha_1(t)$ is in the second term in (2.95), and where $Q := 0$ when $k=0$. Note that Q depends on the form degree k but as is customary we omit this explicit dependence. In local coordinates $(\mathcal{U}, x)$ as considered in (2.96), we have

$$Q\alpha = \sum_J \left(\int_0^1 b_J(x,t)\, dt\right) dx_J. \tag{2.97}$$

Let $i_0 : M \to M\times\mathbb{R}$ be the inclusion $i_0(p) := (p,0)$ for all $p\in M$ and let $i_1 : M\to M\times\mathbb{R}$ be the inclusion $i_1(p) = (p,1)$ for all $p\in M$. Then the operator Q has the following property.

Theorem 2.50 (**Homotopy operator theorem**). *On $C^\infty(M\times\mathbb{R},\Lambda^k)$, we have*

$$\boxed{d\circ Q + Q\circ d = i_1^* - i_0^*.}$$

That is, for all $\alpha\in C^\infty(M\times\mathbb{R},\Lambda^k)$ we have

$$d(Q\alpha) + Q(d\alpha) = i_1^*\alpha - i_0^*\alpha.$$

Proof. For clarity, let us denote by "d" the exterior derivative on the space $C^\infty(M\times\mathbb{R},\Lambda)$ and "d_M" the exterior derivative on $C^\infty(M,\Lambda)$, so that given $\alpha\in C^\infty(M\times\mathbb{R},\Lambda^k)$ we need to show that

$$d_M(Q\alpha) + Q(d\alpha) = i_1^*\alpha - i_0^*\alpha.$$

As we did in (2.95), write

$$\alpha = \alpha_0(t) + dt\wedge\alpha_1(t).$$

Since $i_1(p) = (p,1)$ and $i_0(p) = (p,0)$ for all $p\in M$ it follows that

$$i_1^*\alpha = \alpha_0(1) + (d1)\wedge\alpha_1(1) = \alpha_0(1),$$

since $d(\text{constant}) = 0$, and similarly, $i_0^*\alpha = \alpha_0(0)$. Hence, we have to show that

$$d_M(Q\alpha) + Q(d\alpha) = \alpha_0(1) - \alpha_0(0).$$

To see this, first observe that

$$d_M(Q\alpha) = d_M\left(\int_0^1 \alpha_1(t)\,dt\right) = \int_0^1 d_M\alpha_1(t)\,dt; \tag{2.98}$$

that is, taking d_M just takes the exterior derivative with respect to the x variables in (2.97). Second, observe that

$$d\alpha_0(t) = d_M\alpha_0(t) + dt\wedge(\partial_t\alpha_0(t))$$

and

$$d(dt\wedge\alpha_1(t)) = -dt\wedge(d_M\alpha_1(t)).$$

Indeed, to prove the first formula, write $\alpha_0(t)$ locally in coordinates as we did below (2.96) and compute:

$$\begin{aligned} d\alpha_0(t) = d\left(\sum_I a_I(x,t)\,dx_I\right) &= \sum_I \partial_t a_I(x,t)\,dt\wedge dx_I + \sum_I d_M a_I(x,t)\wedge dx_I \\ &= dt\wedge(\partial_t\alpha_0(t)) + d_M\alpha_0(t). \end{aligned}$$

To prove the second formula we use the first formula with $\alpha_0(t)$ replaced by $\alpha_1(t)$, the anti-derivation property of d, and that $d^2 = 0$:

$$\begin{aligned} d(dt\wedge\alpha_1(t)) &= (d^2t)\wedge\alpha_1(t) - dt\wedge d\alpha_1(t) \\ &= 0 - dt\wedge\big(dt\wedge(\partial_t\alpha_0(t)) + d_M\alpha_0(t)\big) \\ &= -dt\wedge(d_M\alpha_1(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} d\alpha &= d\big(\alpha_0(t) + dt\wedge\alpha_1(t)\big) \\ &= d_M\alpha_0(t) + dt\wedge(\partial_t\alpha_0(t)) - dt\wedge(d_M\alpha_1(t)) \\ &= d_M\alpha_0(t) + dt\wedge(\partial_t\alpha_0(t) - d_M\alpha_1(t)). \end{aligned}$$

Now using the definition of Q and the fundamental theorem of calculus we get

$$Q(d\alpha) = \int_0^1 (\partial_t\alpha_0(t) - d_M\alpha_1(t))\,dt = \alpha_0(1) - \alpha_0(0) - \int_0^1 d_M\alpha_1(t)\,dt.$$

Adding this to (2.98) we get our result. □

2.9.2. Homotopy theorems. Using the homotopy operator theorem we can prove that homotopic maps induce the same map on cohomology.

LEMMA 2.51. *Homotopic maps are the same on cohomology: If $f \simeq g$, then $f^* = g^*$ on de Rham cohomology.*

PROOF. Let $f : M \to N$ and $g : M \to N$ be homotopic; we need to show that $f^* = g^*$ as maps from $H^k(N) \to H^k(M)$ for any k. Let $H : M\times\mathbb{R} \to N$ be a homotopy map with $H(\cdot,0) = f$ and $H(\cdot,1) = g$ and compose the identity $d\circ Q + Q\circ d = i_1^* - i_0^*$ with the pullback H^*:

$$d\circ QH^* + Q\circ dH^* = i_1^*H^* - i_0^*H^* = (H\circ i_1)^* - (H\circ i_0)^*.$$

Since $i_0(p) = (p,0)$ we have $(H\circ i_0)(p) = H(p,0) = f(p)$ and since $i_1(p) = (p,1)$ we have $(H\circ i_1)(p) = H(p,1) = g(p)$. Hence,

$$d\circ QH^* + Q\circ dH^* = g^* - f^*.$$

In particular, applying this identity to a closed form $\alpha \in C^\infty(N, \Lambda^k)$ and using that $dH^*\alpha = H^*(d\alpha) = H^*(0) = 0$, we see that

$$d(QH^*\alpha) = g^*\alpha - f^*\alpha.$$

Hence, $g^*[\alpha] = f^*[\alpha]$, and therefore $f^* = g^*$ as maps from $H^k(N) \to H^k(M)$. □

Manifolds M and N are said to be (smoothly) **homotopy equivalent** if there are map $f : M \to N$ and $g : N \to M$ such that $f \circ g \simeq \mathrm{Id}_N$ and $g \circ f \simeq \mathrm{Id}_M$. Intuitively speaking M and N can be "deformed" into the other.

THEOREM 2.52 (**Homotopy invariance**). *Homotopy equivalent manifolds have isomorphic de Rham cohomologies.*

PROOF. If $f : M \to N$ and $g : N \to M$ are such that $f \circ g \simeq \mathrm{Id}_N$ and $g \circ f \simeq \mathrm{Id}_M$, then by our lemma, for any k we have

$$(f \circ g)^* = g^* \circ f^* = \mathrm{Id} : H^k(N) \to H^k(N)$$

and

$$(g \circ f)^* = f^* \circ g^* = \mathrm{Id} : H^k(M) \to H^k(M).$$

This shows that $f^* : H^k(N) \to H^k(M)$ and $g^* : H^k(M) \to H^k(N)$ are inverses. □

Example 2.44. We claim that $\mathbb{P} = \mathbb{R}^2 \setminus \{0\}$ and $\mathbb{S}^1$ are homotopy equivalent (this should be "intuitively obvious"). Indeed, define $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$ by $f(x) := x/|x|$ and $g : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$ by $g(x) = x$. Then $(f \circ g)(x) = x$ for all $x \in \mathbb{S}^1$ is the identity while $(g \circ f)(x) = x/|x|$ for all $x \in \mathbb{P}$. Thus, we just have to show that $g \circ f$ is homotopic to the identity. However, this is easy: Define $H : \mathbb{P} \times \mathbb{R} \to \mathbb{P}$ by

$$H(x, t) := xe^{-t \log \|x\|},$$

then H is smooth, $H(x, 0) = x$ and $H(x, 1) = x/\|x\|$. Therefore, $\mathbb{P}$ and $\mathbb{S}^1$ have the same de Rham cohomologies and hence, as we know all the cohomologies of $\mathbb{S}^1$ (Example 2.40) we get all the cohomology spaces of $\mathbb{P}$ for free:

$$H^k(\mathbb{P}) = H^k(\mathbb{S}^1) = \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & k > 1. \end{cases}$$

Of course, we already knew this for $k = 0, 1$ and $k > 2$, only $k = 2$ is new.

A manifold M is said to be (smoothly) **contractible (to a point)** if M to homotopy equivalent to a point (a connected zero-dimensional manifold $\{q\}$). Then by homotopy invariance, we have

THEOREM 2.53 (**Poincaré's lemma**). *If M is contractible, then*

$$H^k(M) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

Poincaré's lemma should probably be called Volterra's lemma after Vito Volterra (1860– 1940) since he discovered it first [**8**]. We are actually stating Poincaré's lemma in a more general form; it's usually stated for star-shaped regions.

Example 2.45. We say that an open subset $S \subseteq \mathbb{R}^n$ is **star-shaped** with respect to a point $q \in S$ if for each point $p \in S$, the line segment joining p and q is entirely contained in S; see Figure 2.20. Then S is contractible to the point q. To prove this, define $f : S \to \{q\}$ by $f(x) := q$ and $g : \{q\} \to S$ by $g(q) = q$. Then

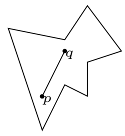


FIGURE 2.20. A star-shaped region in $\mathbb{R}^2$.

$(f \circ g)(x) = x$ for all $x \in \{q\}$ is the identity while $(g \circ f)(x) = q$ for all $x \in S$. Thus, we just have to show that $g \circ f$ is homotopic to the identity. To do so, let $g : \mathbb{R} \to \mathbb{R}$ be the function in Figure 2.1 of Section 2.1 with $\varepsilon = 1$ in Figure 2.1. The function g satisfies $0 \leq g \leq 1$, g is nondecreasing, $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t \geq 1$. Define $H : S \times \mathbb{R} \to S$ by

$$H(x,t) := (1 - g(t))x + g(t)q;$$

then H is smooth, $H(x,0) = x$ and $H(x,1) = q$. (Note that H has range contained in S because S is star-shaped with respect to q by assumption.) Therefore, we have proved that any star-shaped region is contractible. In particular, since $\mathbb{R}^n$ itself is star-shaped, we have

$$\boxed{H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}}$$

In Examples 2.44 and 2.45 we are actually dealing with **deformation retractions**; see Problem 1.

2.9.3. Prelude to the Mayer-Vietoris sequence. We now come to a very powerful and also very straightforward-to-use tool, the Mayer-Vietoris sequence, which can compute de Rham spaces for many familiar spaces by breaking up the spaces into easier to handle ones. The Mayer-Vietoris sequence might seem intimidating to those who aren't comfortable with long exact sequences, so to illustrate the main ideas behind the Mayer-Vietoris sequence, let us compute the de Rham cohomology spaces $H^k(\mathbb{S}^n)$ of the n-sphere $\mathbb{S}^n$ using Mayer-Vietoris techniques but *without* actually using the Mayer-Vietoris sequence. We already know these spaces for $n = 1$ so assume that $n > 1$. We shall prove that

$$H^1(\mathbb{S}^n) = 0 \quad \text{for all } n > 1, \tag{2.99}$$

and

$$H^k(\mathbb{S}^n) \cong H^{k-1}(\mathbb{S}^{n-1}) \quad \text{for all } k > 1 \text{ and } n > 1. \tag{2.100}$$

Using (2.99) and (2.100) we can compute all the de Rham cohomology spaces of $\mathbb{S}^n$. Indeed, we already know $H^0(\mathbb{S}^n) \cong \mathbb{R}$ and $H^k(\mathbb{S}^n) = 0$ for $k > n$, and by (2.99) we have $H^1(\mathbb{S}^n) = 0$, so assume that $2 \leq k \leq n$. Then by (2.100), we have

$$\begin{aligned} H^k(\mathbb{S}^n) \cong H^{k-1}(\mathbb{S}^{n-1}) \cong H^{k-2}(\mathbb{S}^{n-2}) \cong \cdots \cong H^1(\mathbb{S}^{n-k+1}) \\ \cong \begin{cases} \mathbb{R} & k = n \text{ (since } H^1(\mathbb{S}^1) \cong \mathbb{R}) \\ 0 & 2 \leq k < n \text{ (by (2.99)).} \end{cases} \end{aligned}$$

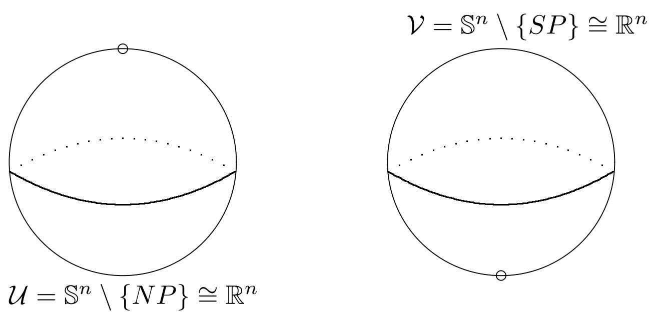


FIGURE 2.21. The open sets $\mathcal{U}$ and $\mathcal{V}$.

In conclusion, given (2.99) and (2.100) we have proved that

$$\boxed{H^k(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n. \end{cases}}$$

Now to the proofs of (2.99) and (2.100). They are both based on the following "trick" of writing $\mathbb{S}^n$ as the union of two sets both of which are diffeomorphic to $\mathbb{R}^n$ (and hence have easy cohomology spaces). Let $\mathcal{U} = \mathbb{S}^n \setminus \{NP\}$, where $NP = (0, 0, \ldots, 0, 1)$ is the "north pole" and $\mathcal{V} = \mathbb{S}^n \setminus \{SP\}$ where $SP = (0, 0, \ldots, 0, -1)$ is the south pole; see Figure 2.21.

Let's prove (2.99): $H^1(\mathbb{S}^n) = 0$, which is to say, if $\alpha \in C^\infty(\mathbb{S}^n, \Lambda^1)$ and $d\alpha = 0$, then $\alpha = df$ for some $f \in C^\infty(\mathbb{S}^n, \mathbb{R})$. To prove this, observe that since $\mathcal{U} \cong \mathbb{R}^n$ by Poincaré's lemma we know that

$$\alpha|_{\mathcal{U}} = dg \quad \text{for some} \quad g \in C^\infty(\mathcal{U}, \mathbb{R}),$$

and similarly

$$\alpha|_{\mathcal{V}} = dh \quad \text{for some} \quad h \in C^\infty(\mathcal{V}, \mathbb{R}),$$

Therefore, on the intersection $\mathcal{U} \cap \mathcal{V}$, we have

$$d(g - h) = dg - dh = \alpha - \alpha = 0 \quad \text{on} \quad \mathcal{U} \cap \mathcal{V}.$$

Thus, as $\mathcal{U} \cap \mathcal{V}$ is connected (see e.g. Figure 2.22 below for a picture of $\mathcal{U} \cap \mathcal{V}$) it follows that $g = h + c$ for a constant $c \in \mathbb{R}$. Now define $f : \mathbb{S}^n \to \mathbb{R}$ by

$$f(p) := \begin{cases} g(p) & \text{if } p \in \mathcal{U} \\ h(p) + c & \text{if } p \in \mathcal{V}. \end{cases}$$

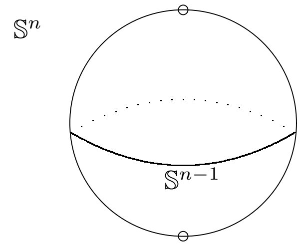
Since $g = h + c$ on $\mathcal{U} \cap \mathcal{V}$, f is well-defined on $\mathcal{U} \cap \mathcal{V}$ and therefore $f \in C^\infty(\mathbb{S}^n, \mathbb{R})$ since both g and h are smooth. Moreover, $df = dg = \alpha$ on $\mathcal{U}$ and $df = d(h + c) = dh = \alpha$ on $\mathcal{V}$ and hence $df = \alpha$ on all of $\mathbb{S}^n$. This proves (2.99).

Now to prove (2.100): $H^k(\mathbb{S}^n) \cong H^{k-1}(\mathbb{S}^{n-1})$ for $k > 1$. We shall construct an isomorphism from $H^k(\mathbb{S}^n)$ to $H^{k-1}(\mathbb{S}^{n-1})$. To do so, let $[\alpha] \in H^k(\mathbb{S}^n)$. Since $\mathcal{U}$ and $\mathcal{V}$ are both diffeomorphic to $\mathbb{R}^n$, by Poincaré's lemma we have

$$\alpha = d\beta \;\text{ on }\; \mathcal{U} \quad \text{and} \quad \alpha = d\gamma \;\text{ on }\; \mathcal{V}, \tag{2.101}$$

where $\beta \in C^\infty(\mathcal{U}, \Lambda^{k-1})$ and $\gamma \in C^\infty(\mathcal{V}, \Lambda^{k-1})$. On the intersection $\mathcal{U} \cap \mathcal{V}$, we have

$$d(\beta - \gamma) = d\beta - d\gamma = \alpha - \alpha = 0 \quad \text{on} \quad \mathcal{U} \cap \mathcal{V},$$



$$\mathbb{S}^n \setminus \{NP, SP\} \cong \mathbb{S}^{n-1} \times (-1,1)$$

FIGURE 2.22. The intersection $\mathcal{U} \cap \mathcal{V} = \mathbb{S}^n \setminus \{NP, SP\}$.

hence $\beta - \gamma$ is closed on $\mathcal{U} \cap \mathcal{V}$ and hence $[\beta - \gamma] \in H^{k-1}(\mathcal{U} \cap \mathcal{V})$. Therefore, we have a map

$$H^k(\mathbb{S}^n) \ni [\alpha] \;\mapsto\; [\beta - \gamma] \in H^{k-1}(\mathcal{U} \cap \mathcal{V}). \tag{2.102}$$

There are choices we made to define this map. Namely, we could have chosen different forms β and γ satisfying (2.101) (the new forms can only differ from β and γ by exact forms) and even at the very beginning we could have chosen another element of the class $[\alpha]$ instead of α (the form can only differ from α by an exact form). However, we leave it as an (not difficult) exercise to prove that the class $[\beta - \gamma] \in H^{k-1}(\mathcal{U} \cap \mathcal{V})$ is independent of the choices made. Therefore, the map (2.102) is well-defined. We shall prove that the map (2.102) is in fact an isomorphism. This shows that

$$H^k(\mathbb{S}^n) \cong H^{k-1}(\mathcal{U} \cap \mathcal{V}).$$

Now as seen in Figure 2.22 (and is also easily proved) that $\mathcal{U} \cap \mathcal{V} \cong \mathbb{S}^{n-1} \times (-1,1)$. It is obvious that $\mathbb{S}^{n-1} \times (-1,1)$ is homotopy equivalent to $\mathbb{S}^{n-1} \times \{0\} \cong \mathbb{S}^{n-1}$ (just shrink $(-1,1)$ to $\{0\}$). Therefore,

$$H^k(\mathbb{S}^n) \cong H^{k-1}(\mathcal{U} \cap \mathcal{V}) \cong H^{k-1}(\mathbb{S}^{n-1}),$$

and (2.100) is proved. Thus, it remains now to prove that (2.102) is an isomorphism. To do so we define an inverse map taking and element of $H^{k-1}(\mathcal{U}\cap\mathcal{V})$ to an element of $H^k(\mathbb{S}^n)$. We define this map as follows. Choose a partition of unity $\{\varphi, \psi\}$ subordinate to the cover $\{\mathcal{U}, \mathcal{V}\}$ of $\mathbb{S}^n$; thus,

$$0 \le \varphi, \psi \le 1\,,\ \operatorname{supp} \varphi \subseteq \mathcal{U}\,,\ \operatorname{supp} \psi \subseteq \mathcal{V}\,,\ \varphi + \psi = 1.$$

See Figure 2.23 for an example of some properties of such a partition of unity.

Now, given $[\xi] \in H^{k-1}(\mathcal{U} \cap \mathcal{V})$ with $\xi \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^{k-1})$ a closed form, we map it to

$$[d\psi \wedge \xi] \in H^k(\mathbb{S}^n) \quad \text{where} \;\; d\psi \wedge \xi \in C^\infty(\mathbb{S}^n, \Lambda^k).$$

Here, we note that $d\psi$ is supported in $\mathcal{U} \cap \mathcal{V}$ as seen in Figure 2.23. Also, $\xi \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^{k-1})$ so we can extend $d\psi \wedge \xi$ by zero to the rest of $\mathbb{S}^n$ and we get a smooth form on all of $\mathbb{S}^n$. Finally, we note that since ξ is closed, so is $d\psi \wedge \xi$:

$$d\big(d\psi \wedge \xi\big) = d(d\psi) \wedge \xi - d\psi \wedge d\xi = 0,$$

Therefore, we have defined a map

$$H^{k-1}(\mathcal{U} \cap \mathcal{V}) \ni [\xi] \;\mapsto\; [d\psi \wedge \xi] \in H^k(\mathbb{S}^n). \tag{2.103}$$

It's easy to check that this map is well-defined, independent of the choice of representative of the class of ξ. Finally, we are left to prove that the map (2.103) is the inverse of the map (2.102). We state this as a lemma.

LEMMA 2.54. *The maps* (2.103) *and* (2.102) *are inverses.*

PROOF. We prove this in two steps.

Step 1: We first prove that (2.102) $\circ$ (2.103) = Id. Let $[\xi] \in H^{k-1}(\mathcal{U} \cap \mathcal{V})$ and map it to $[\alpha := d\psi \wedge \xi] \in H^k(\mathbb{S}^n)$. We now apply the map (2.102) to $[\alpha]$ and show that we get $[\xi]$. Indeed, over $\mathcal{U}$ we have

$$\text{On } \mathcal{U}, \ \ \alpha = d\psi \wedge \xi = d\beta, \quad \text{where } \ \beta := \psi\xi \in C^\infty(\mathcal{U}, \Lambda^{k-1}).$$

Note that ξ is really only defined on $\mathcal{U} \cap \mathcal{V} = \mathbb{S}^n \setminus \{NP, SP\}$. However, $\psi \equiv 0$ near SP, therefore $\psi\xi$ (by defining it to be 0 near SP) is a smooth form on $\mathbb{S}^n \setminus \{NP\} = \mathcal{U}$. This is why $\beta \in C^\infty(\mathcal{U}, \Lambda^{k-1})$.

Now over $\mathcal{V}$ we have, using that $d\varphi = -d\xi$ (since $\varphi + \psi = 1$),

$$\text{On } \mathcal{V}, \ \ \alpha = d\psi \wedge \xi = -d\varphi \wedge \xi = d\gamma, \quad \text{where } \ \gamma := -\varphi\xi \in C^\infty(\mathcal{V}, \Lambda^{k-1}).$$

As above, ξ is really only defined on $\mathcal{U} \cap \mathcal{V} = \mathbb{S}^n \setminus \{NP, SP\}$, however, $\varphi \equiv 0$ near NP, therefore $\varphi\xi$ (by defining it to be 0 near NP) is a smooth form on $\mathbb{S}^n \setminus \{SP\} = \mathcal{V}$. This is why $\gamma \in C^\infty(\mathcal{V}, \Lambda^{k-1})$.

Observe that since $\varphi + \psi = 1$,

$$\text{On } \mathcal{U} \cap \mathcal{V}, \ \ \beta - \gamma = \psi\xi - (-\varphi\xi) = \psi\xi + \varphi\xi = \xi.$$

Therefore, by definition of the map (2.102), we have

$$[\alpha] \mapsto [\beta - \gamma] = [\xi].$$

Step 2: Next, we prove that (2.103) $\circ$ (2.102) = Id. Let $[\alpha] \in H^k(\mathbb{S}^n)$ and map it to $[\beta - \gamma] \in H^k(\mathcal{U} \cap \mathcal{V})$. We now apply the map (2.103) to $[\beta - \gamma]$ and show that we get $[\alpha]$ again. By definition, we have

$$[\beta - \gamma] \mapsto [d\psi \wedge (\beta - \gamma)] = [d\psi \wedge \beta - d\psi \wedge \gamma].$$

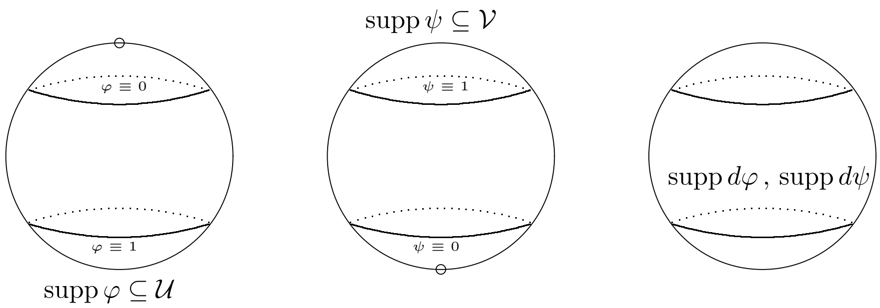


FIGURE 2.23. The partition of unity $\{\varphi, \psi\}$ of $\mathbb{S}^n$ subordinate to the cover $\{\mathcal{U}, \mathcal{V}\}$. Note that since $\varphi + \psi = 1$, we have $d\varphi = -d\psi$.

Now observe that

$$\begin{aligned} d\psi \wedge \beta - d\psi \wedge \gamma &= -d\varphi \wedge \beta - d\psi \wedge \gamma \qquad (d\varphi = -d\psi) \\ &= -d(\varphi\beta) + \varphi d\beta - d(\psi\gamma) + \psi d\gamma \\ &= -d(\varphi\beta) + \varphi\alpha - d(\psi\gamma) + \psi\alpha \qquad (\alpha = d\beta,\, \alpha = d\gamma) \\ &= \alpha - d\big(\varphi\beta + \psi\gamma\big) \qquad (\varphi + \psi = 1). \end{aligned}$$

Note that since $\operatorname{supp}\varphi \subseteq \mathcal{U}$ and $\beta \in C^\infty(\mathcal{U}, \Lambda^{k-1})$ we have $\varphi\beta \in C^\infty(\mathbb{S}^n, \Lambda^{k-1})$ by extending $\varphi\beta$ to be zero off of $\mathcal{U}$. Similarly $\psi\gamma \in C^\infty(\mathbb{S}^n, \Lambda^{k-1})$. Therefore, $\varphi\beta + \psi\gamma \in C^\infty(\mathbb{S}^n, \Lambda^{k-1})$. Therefore,

$$[d\psi \wedge \beta - d\psi \wedge \gamma] = [\alpha],$$

and we are done. □

2.9.4. Mayer-Vietoris sequence. Using basically the same techniques we used to compute $H^k(\mathbb{S}^n)$, we introduce the Mayer-Vietoris sequence. First of all, recall that a (finite or infinite) sequence of vector spaces and linear maps

$$\cdots V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} V_4 \xrightarrow{f_4} V_5 \cdots$$

is said to be **exact** at V_j if $\ker f_j = \operatorname{Im} f_{j-1}$ (that is, if the kernel of the map with domain V_j is equal to the image of the map with codomain V_j). The sequence is **exact** if it is exact at each V_j.

An exact sequence of the form

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is called a **short exact sequence**. Here the first map $0 \to U$ is the inclusion $0 \to 0$ and the last map $W \to 0$ is the map that takes everything in W to 0. A short exact sequence can be thought of as follows. First, exactness at U means that $0 = \ker f$, which is to say f is injective, so $U \cong \operatorname{Im} f$ and therefore we can consider U as a subspace of V by identifying U with $\operatorname{Im} f$. In this way, the sequence becomes

$$0 \to U \xrightarrow{\iota} V \xrightarrow{g} W \to 0,$$

where ι is inclusion. Exactness at W means that $\operatorname{Im} g = \ker(W \to 0) = W$, therefore g is surjective and exactness at V means that $\ker g = \operatorname{Im} \iota = U$. Hence,

$$W \cong V/\ker g = V/U.$$

Therefore, a short exact sequence can be thought of as a sequence of the form

$$0 \to U \xrightarrow{\iota} V \xrightarrow{\pi} V/U \to 0$$

where $\pi : V \to V/U$ is the projection $v \mapsto [v]$. The most important example of a short exact sequence for us is the following.

Example 2.46. Let M be a manifold and write $M = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}, \mathcal{V} \subseteq M$ are two open sets. Define the restriction map

$$r : C^\infty(M, \Lambda^k) \to C^\infty(\mathcal{U}, \Lambda^k) \oplus C^\infty(\mathcal{V}, \Lambda^k)$$

by

$$r(\alpha) := (\alpha|_{\mathcal{U}}, \alpha|_{\mathcal{V}}) \in C^\infty(\mathcal{U}, \Lambda^k) \oplus C^\infty(\mathcal{V}, \Lambda^k).$$

Define the "minus" map

$$m : C^\infty(\mathcal{U}, \Lambda^k) \oplus C^\infty(\mathcal{V}, \Lambda^k) \to C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k)$$

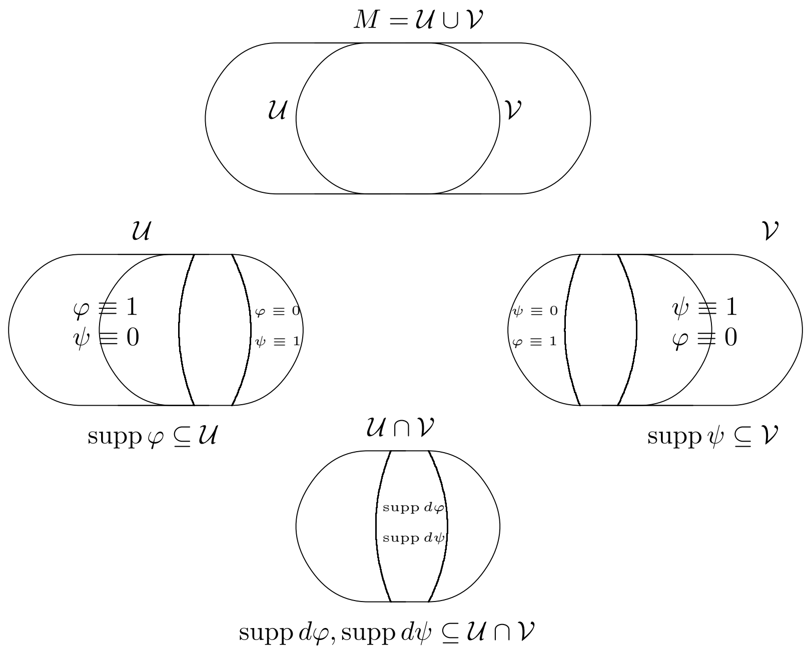


FIGURE 2.24. The open sets (ovals) $\mathcal{U}$ and $\mathcal{V}$, $M = \mathcal{U} \cup \mathcal{V}$ (overlapping ovals), and the partition of unity φ and ψ.

by

$$m(\beta, \gamma) := \beta - \gamma \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k),$$

where the difference $\beta - \gamma$ is understood to have the domain $\mathcal{U} \cap \mathcal{V}$, the intersection of the domains of β and γ. We claim that the following sequence

$$\boxed{0 \to C^\infty(M, \Lambda^k) \xrightarrow{r} C^\infty(\mathcal{U}, \Lambda^k) \oplus C^\infty(\mathcal{V}, \Lambda^k) \xrightarrow{m} C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k) \to 0} \tag{2.104}$$

is short exact. Indeed, exactness at $C^\infty(M, \Lambda^k)$ (that is, injectivity of r) simply means that if a k-form on M vanishes on both $\mathcal{U}$ and $\mathcal{V}$, then it vanishes on all of M. Exactness at the middle term $C^\infty(\mathcal{U}, \Lambda^k) \oplus C^\infty(\mathcal{V}, \Lambda^k)$ (that is, $\ker m = \operatorname{Im} r$ just means that a k-form β on $\mathcal{U}$ and a k-form γ on $\mathcal{V}$ are equal on $\mathcal{U} \cap \mathcal{V}$ (that is, $\beta - \gamma = 0$ on $\mathcal{U} \cap \mathcal{V}$ or $(\beta, \gamma) \in \ker m$) if and only if β and γ patch together to define a k-form α on all of $M = \mathcal{U} \cup \mathcal{V}$ (that is $r(\alpha) = (\beta, \gamma)$ or $(\beta, \gamma) \in \operatorname{Im} r$). Exactness at $C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k)$ is the hardest. We need to show that m is surjective, so let ξ be a k-form on $\mathcal{U} \cap \mathcal{V}$. Choose a partition of unity $\{\varphi, \psi\}$ subordinate to the cover $\{\mathcal{U}, \mathcal{V}\}$ of M (see Figure 2.24):

$$0 \leq \varphi, \psi \leq 1\,,\ \operatorname{supp} \varphi \subseteq \mathcal{U}\,,\ \operatorname{supp} \psi \subseteq \mathcal{V}\,,\ \varphi + \psi = 1.$$

Let $\beta := \psi\xi$. Since ξ is defined on $\mathcal{U} \cap \mathcal{V}$ and $\psi|_{\mathcal{U}}$ is smooth on $\mathcal{U}$ that vanishes outside of $\mathcal{U} \cap \mathcal{V}$ (see the middle-left picture in Figure 2.24), we can extend $\psi\xi$ by zero so that it is a smooth k-form on $\mathcal{U}$. Similarly, $\gamma := -\varphi\xi$ is a smooth k-form on $\mathcal{V}$. Hence,

$$\boxed{\begin{aligned} &\text{Useful fact: } \xi \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k) \implies \\ &\qquad\qquad (\psi\xi, -\varphi\xi) \in C^\infty(\mathcal{U}, \Lambda^k) \oplus C^\infty(\mathcal{V}, \Lambda^k). \end{aligned}} \tag{2.105}$$

Then, since $\varphi + \psi = 1$, we have

$$m(\psi\xi, -\varphi\xi) = \psi\xi + \varphi\xi = (\psi + \varphi)\xi = \xi.$$

Therefore m is surjective and hence (2.104) is exact.

One can check that r and m map closed forms to closed forms and exact forms to exact forms and hence they induce maps on cohomology

$$H^k(M) \xrightarrow{r} H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) \xrightarrow{m} H^k(\mathcal{U} \cap \mathcal{V}).$$

We define a map

$$\delta : H^k(\mathcal{U} \cap \mathcal{V}) \to H^{k+1}(M)$$

by copying the construction in (2.103) used for $\mathbb{S}^n$. Namely, for a fixed partition of unity of M as in Figure 2.24, given $[\xi] \in H^k(\mathcal{U} \cap \mathcal{V})$ with $\xi \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k)$ a closed form, we map it to

$$\delta[\xi] := [d\psi \wedge \xi] \in H^{k+1}(M) \quad \text{where} \quad d\psi \wedge \xi \in C^\infty(M, \Lambda^{k+1}).$$

We remark since that $d\psi$ is supported in $\mathcal{U} \cap \mathcal{V}$ as seen in Figure 2.24 and $\xi \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k)$, we can extend $d\psi \wedge \xi$ by zero to the rest of M to get a smooth form on all of M. Also, we remark that since ξ is closed, so is $d\psi \wedge \xi$:

$$d\big(d\psi \wedge \xi\big) = d(d\psi) \wedge \xi - d\psi \wedge d\xi = 0.$$

Finally, it's easy to check the map δ is well-defined, independent of the choice of representative of the class of ξ. One can even check that δ is defined independent of the choice of partition of unity $\{\varphi, \psi\}$ subordinate to $\{\mathcal{U}, \mathcal{V}\}$. Any case, combining the maps r, m, δ, we get a sequence of vector spaces and maps, called the **Mayer-Vietoris sequence**:

$$\begin{array}{ccccccc}
H^{k+1}(M) & \xrightarrow{r} & H^{k+1}(\mathcal{U}) \oplus H^{k+1}(\mathcal{V}) & \xrightarrow{m} & H^{k+1}(\mathcal{U} \cap \mathcal{V}) \to \cdots & & \\
 & & & \delta & & & \\
H^1(M) & & \to \cdots \to H^k(M) & \xrightarrow{r} & H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) & \xrightarrow{m} & H^k(\mathcal{U} \cap \mathcal{V}) \\
 & & & \delta & & & \\
0 & \longrightarrow & H^0(M) & \xrightarrow{r} & H^0(\mathcal{U}) \oplus H^0(\mathcal{V}) & \xrightarrow{m} & H^0(\mathcal{U} \cap \mathcal{V}).
\end{array}$$

The Mayer-Vietoris theorem below states that this sequence is exact. Tools from homological algebra (e.g. the notorious "Zig-zag lemma") using commutative diagrams, diagram chasing, and abstract nonsense, one can use the exact sequence (2.104) of Example 2.46 to derive the Mayer-Vietoris theorem. However, one of the (many) beautiful aspects about differential forms is their concreteness: One can prove the Mayer-Vietoris theorem "directly" without the knowledge of homological algebra nor with drawing a single commutative diagram! We shall prove the Mayer-Vietoris theorem directly leaving the homological algebra proof to you.

THEOREM 2.55 (**Mayer-Vietoris theorem**). *The sequence*

$$\boxed{0 \to \cdots \to H^k(M) \xrightarrow{r} H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) \xrightarrow{m} H^k(\mathcal{U} \cap \mathcal{V}) \xrightarrow{\delta} H^{k+1}(M) \to \cdots}$$

is exact.

PROOF. The "hardest" parts to prove are exactness at $H^k(\mathcal{U}\cap\mathcal{V})$ and $H^{k+1}(M)$ since these involve the map δ, so we shall prove exactness at these places leaving exactness at the other places for you.

Exactness at $H^k(\mathcal{U}\cap\mathcal{V})$*:*

$$H^k(\mathcal{U})\oplus H^k(\mathcal{V}) \xrightarrow{m} H^k(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\delta} H^{k+1}(M)$$

We need to show that (i) $\operatorname{Im} m \subseteq \ker\delta$ and (ii) $\ker\delta \subseteq \operatorname{Im} m$. To prove (i), let $([\beta],[\gamma]) \in H^k(\mathcal{U})\oplus H^k(\mathcal{V})$. Then,

$$\delta(m([\beta],[\gamma])) = \delta[\beta-\gamma] = [d\psi\wedge(\beta-\gamma)].$$

Observe that

$$\begin{aligned} d\psi\wedge(\beta-\gamma) = d\psi\wedge\beta - d\psi\wedge\gamma &= -d\varphi\wedge\beta - d\psi\wedge\gamma \quad (\text{since } d\psi = -d\varphi)\\ &= -d(\varphi\beta) - d(\psi\gamma) \quad (\text{since } d\beta = 0 \text{ and } d\gamma = 0)\\ &= -d(\varphi\beta+\psi\gamma). \end{aligned}$$

Now, $\operatorname{supp}\varphi \subseteq \mathcal{U}$ and $\beta \in C^\infty(\mathcal{U},\Lambda^k)$ so $\varphi\beta$ can be extended by zero to define $\varphi\beta \in C^\infty(M,\Lambda^k)$. Similarly, $\operatorname{supp}\psi \subseteq \mathcal{V}$ and $\gamma \in C^\infty(\mathcal{V},\Lambda^k)$ so $\psi\gamma$ can be extended by zero to define $\psi\gamma \in C^\infty(M,\Lambda^k)$. This shows that $d\psi\wedge(\beta-\gamma) \in C^\infty(M,\Lambda^k)$ is exact. Hence,

$$\delta(m([\beta],[\gamma])) = [d\psi\wedge(\beta-\gamma)] = 0.$$

We now prove (ii) $\ker\delta \subseteq \operatorname{Im} m$. Let $[\xi] \in H^k(\mathcal{U}\cap\mathcal{V})$ and assume that

$$\delta[\xi] := [d\psi\wedge\xi] = 0 \in H^{k+1}(M),$$

which means that $d\psi\wedge\xi = d\alpha$ for some $\alpha \in C^\infty(M,\Lambda^k)$; we need to show that $[\xi] \in \operatorname{Im} m$. Consider

$$(\psi\xi-\alpha, -\varphi\xi-\alpha) \in C^\infty(\mathcal{U},\Lambda^k)\oplus C^\infty(\mathcal{V},\Lambda^k),$$

where as in (2.105) we can regard $\psi\xi \in C^\infty(\mathcal{U},\Lambda^k)$ and $\varphi\xi \in C^\infty(\mathcal{V},\Lambda^k)$ and where α on the left (resp. right) factor is understood to be $\alpha|_{\mathcal{U}}$ (resp. $\alpha|_{\mathcal{V}}$). Observe that

$$d(\psi\xi-\alpha) = d\psi\wedge\xi - d\alpha = 0,$$

and (using that $d\varphi = -d\psi$)

$$d(-\varphi\xi-\alpha) = -d\varphi\wedge\xi - d\alpha = d\psi\wedge\xi - d\alpha = 0.$$

Therefore, $([\psi\xi-\alpha],[-\varphi\xi-\alpha]) \in H^k(\mathcal{U})\oplus H^k(\mathcal{V})$, and moreover, since $\varphi+\psi = 1$, we have

$$m([\psi\xi-\alpha],[-\varphi\xi-\alpha]) = [\psi\xi-\alpha+\varphi\xi+\alpha] = [\xi].$$

Hence, $\ker\delta \subseteq \operatorname{Im} m$.

Exactness at $H^{k+1}(M)$*:*

$$H^k(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\delta} H^{k+1}(M) \xrightarrow{r} H^{k+1}(\mathcal{U})\oplus H^{k+1}(\mathcal{V}).$$

We need to show that (i) $\operatorname{Im}\delta \subseteq \ker r$ and (ii) $\ker r \subseteq \operatorname{Im}\delta$. To prove (i), let $[\xi] \in H^k(\mathcal{U}\cap\mathcal{V})$. Then,

$$r(\delta[\xi]) = [d\psi\wedge\xi] = ([(d\psi\wedge\xi)|_{\mathcal{U}}]\,,\,[(d\psi\wedge\xi)|_{\mathcal{V}}]).$$

As in (2.105) we can regard $\psi\xi \in C^\infty(\mathcal{U},\Lambda^k)$ and $\varphi\xi \in C^\infty(\mathcal{V},\Lambda^k)$, in which case on $\mathcal{U}$ we have

$$d\psi\wedge\xi = d(\psi\xi) \quad , \quad \text{where } \psi\xi \in C^\infty(\mathcal{U},\Lambda^k),$$

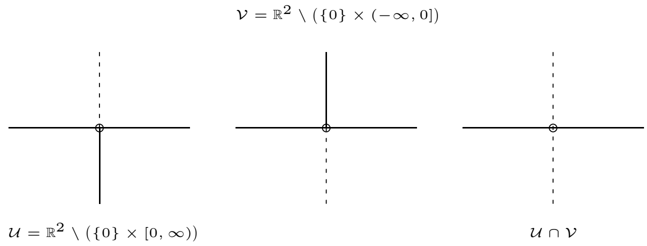


FIGURE 2.25. Writing $\mathbb{P}$ as the union of two sets $\mathcal{U}$ and $\mathcal{V}$.

and, since $d\varphi = -d\psi$, on $\mathcal{V}$ we have

$$d\psi \wedge \xi = -d\varphi \wedge \xi = d(-\varphi\xi) \quad , \quad \text{where } \; -\varphi\xi \in C^\infty(\mathcal{V}, \Lambda^k).$$

Therefore, $(d\psi \wedge \xi)|_{\mathcal{U}}$ and $(d\psi \wedge \xi)|_{\mathcal{V}}$ are exact and hence $r(\delta[\xi]) = 0$. We now prove (ii) $\ker r \subseteq \operatorname{Im} \delta$. Let $[\alpha] \in H^{k+1}(M)$ and assume that

$$r[\alpha] := ([\alpha|_{\mathcal{U}}], [\alpha|_{\mathcal{V}}]) = (0, 0) \in H^{k+1}(\mathcal{U}) \oplus H^{k+1}(\mathcal{V}),$$

which means that

$$\text{On } \mathcal{U}, \;\; \alpha = d\beta, \quad \beta \in C^\infty(\mathcal{U}, \Lambda^k)$$

and

$$\text{On } \mathcal{V}, \;\; \alpha = d\gamma, \quad \gamma \in C^\infty(\mathcal{V}, \Lambda^k).$$

We want to show that $[\alpha] = \delta[\xi] = [d\psi \wedge \xi]$ for some closed $\xi \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^{k-1})$. The "obvious" choice for the closed element is $\xi := \beta - \gamma$, which is closed because

$$\text{on } \mathcal{U} \cap \mathcal{V}, \quad d\xi = d(\beta - \gamma) = d\beta - d\gamma = \alpha - \alpha = 0.$$

To see that this ξ works, observe that

$$\begin{aligned} d\psi \wedge \xi = d\psi \wedge \beta - d\psi \wedge \gamma &= -d\varphi \wedge \beta - d\psi \wedge \gamma \qquad (d\varphi = -d\psi) \\ &= -d(\varphi\beta) + \varphi d\beta - d(\psi\gamma) + \psi d\gamma \\ &= -d(\varphi\beta) + \varphi\alpha - d(\psi\gamma) + \psi\alpha \qquad (\alpha = d\beta, \, \alpha = d\gamma) \\ &= \alpha - d\big(\varphi\beta + \psi\gamma\big) \qquad (\varphi + \psi = 1). \end{aligned}$$

Here, note that since $\operatorname{supp}\psi \subseteq \mathcal{V}$ and $\gamma \in C^\infty(\mathcal{V}, \Lambda^k)$, $\psi\gamma$ can be extended by zero to define $\psi\gamma \in C^\infty(M, \Lambda^k)$. Similarly, $\varphi\beta \in C^\infty(M, \Lambda^k)$. In summary, we have proved that with $\xi := \beta - \gamma \in C^\infty(\mathcal{U} \cap \mathcal{V}, \Lambda^k)$ and $\eta := \varphi\beta + \psi\gamma \in C^\infty(M, \Lambda^k)$, we have on M,

$$\alpha = d\psi \wedge \xi + d\eta \quad \Longrightarrow \quad [\alpha] = [d\psi \wedge \xi] = \delta[\xi].$$

Therefore, $\ker r \subseteq \operatorname{Im} \delta$ and our proof is complete. □

Example 2.47. Let's repeat Example 2.44, but this time using the Mayer-Vietoris sequence to calculate the de Rham cohomology of the punctured plane $\mathbb{P} = \mathbb{R}^2 \setminus \{0\}$; see Problem 3 for generalizations. We shall apply Mayer-Vietoris to the open cover $\mathbb{P} = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are seen in Figure 2.25 and prove that

$$\boxed{H^k(\mathbb{P}) \cong \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & k \neq 0, 1. \end{cases}}$$

Of course, we know that $H^0(\mathbb{P}) = \mathbb{R}$ (since $\mathbb{P}$ is connected) and $H^k(\mathbb{P}) = 0$ for $k > 2$ so we really only need $H^1(\mathbb{P})$ and $H^2(\mathbb{P})$. The latter is easy: Consider the Mayer-Vietoris sequence

$$\cdots \to H^1(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\delta} H^2(\mathbb{P}) \xrightarrow{r} H^2(\mathcal{U}) \oplus H^2(\mathcal{V}) \to \cdots$$

Since $\mathcal{U}$ and $\mathcal{V}$ are contractible, $H^2(\mathcal{U}) \oplus H^2(\mathcal{V}) = 0$, and since $\mathcal{U}\cap\mathcal{V}$ is a union of two contractible sets, say X and Y (the left and right-half planes), it follows that

$$H^1(\mathcal{U}\cap\mathcal{V}) \cong H^1(X) \oplus H^1(Y) = 0 \oplus 0 = 0;$$

cf. Problem 2 in Exercises 2.8. Therefore, the above sequence is

$$\cdots \to 0 \xrightarrow{\delta} H^2(\mathbb{P}) \xrightarrow{r} 0 \to \cdots$$

Exactness here means that

$$H^2(\mathbb{P}) = \ker r = \operatorname{Im}\delta = 0.$$

Now consider $H^1(\mathbb{P})$. We have

$$0 \to H^0(\mathbb{P}) \xrightarrow{r} H^0(\mathcal{U}) \oplus H^0(\mathcal{V}) \xrightarrow{m} H^0(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\delta} H^1(\mathbb{P}) \xrightarrow{r} H^1(\mathcal{U}) \oplus H^1(\mathcal{V}) \to \cdots$$

Since $H^1(\mathcal{U}) = H^1(\mathcal{V}) = 0$, we can write this sequence as

$$0 \to H^0(\mathbb{P}) \xrightarrow{r} H^0(\mathcal{U}) \oplus H^0(\mathcal{V}) \xrightarrow{m} H^0(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\delta} H^1(\mathbb{P}) \xrightarrow{r} 0 \to \cdots \tag{2.106}$$

Exactness at $H^1(\mathbb{P})$ implies that $\operatorname{Im}\delta = H^1(\mathbb{P})$ and the map m here is the map

$$H^0(\mathcal{U}) \oplus H^0(\mathcal{V}) \ni (a,b) \mapsto a - b \in H^0(\mathcal{U}\cap\mathcal{V}), \quad a,b \in \mathbb{R},$$

so it follows that the image of m is one-dimensional. Here, we recall Proposition 2.46 which states that for any manifold M, $H^0(M)$ is just functions that are constant on the connected components of M. Therefore, by the dimension theorem of linear algebra, we have

$$\begin{aligned} 2 = \dim H^0(\mathcal{U}\cap\mathcal{V}) &= \dim\ker\delta + \dim\operatorname{Im}\delta \\ &= \dim\operatorname{Im} m + \dim H^1(\mathbb{P}) \quad (\text{since } \ker\delta = \operatorname{Im} m) \\ &= 1 + \dim H^1(\mathbb{P}). \end{aligned}$$

Therefore, $\dim H^1(\mathbb{P}) = 1$ and we're done.

You might have been wondering if the de Rham cohomology spaces are finite-dimensional. This is true if the manifold is of finite type, which we now define. A manifold M is said to be of **finite type** if it can be covered by finitely many open sets $\mathcal{U}_1, \ldots, \mathcal{U}_m$ such that each $\mathcal{U}_i$ and any nonempty intersection $\mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_k}$ is contractible. Most manifolds that immediately come to mind are of finite type and you really have to think to come up with non finite-type manifolds in "nature" like $\mathbb{R}^n$, $\mathbb{R}^n$ with finitely many points removed, all spheres, tori, etc. (For example, prove that $\mathbb{S}^1$ is of finite type; in fact, one can take $m = 3$ sets in such a cover — try to draw these sets!) Of course, it's not hard to construct "artificial" non-finite type manifolds like

$$M = \bigcup_{i=0}^{\infty} (i-1, i) = (0,1) \cup (1,2) \cup (2,3) \cup \cdots,$$

an infinite union of disjoint open intervals. Notice that $H^0(M)$, which is the space of functions constant on the connected components of M, is infinite-dimensional for the manifold $M = \bigcup_{i=0}^{\infty}(i-1, i)$. When the manifold is of finite type, all the cohomology spaces of the manifold are finite-dimensional. This is a corollary of Mayer-Vietoris.

COROLLARY 2.56 (**Finite-dimensionality**). *If M is of finite type, then $H^*(M)$ is finite-dimensional.*

PROOF. Call a manifold of "finite type m" if the manifold can be covered by m sets $\{\mathcal{U}_1, \ldots, \mathcal{U}_m\}$ such that each $\mathcal{U}_i$ and any nonempty intersection $\mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_k}$ is contractible. We prove this corollary by induction on m. If $m = 1$, then the manifold is contractible so by Poincaré's lemma, $H^k(M)$ is finite-dimensional for any k. Assume that our result holds for all manifolds of finite type m; we shall prove our result for manifolds of finite type $m+1$. Let M be a manifold of finite type $m+1$ with cover $\{\mathcal{U}_1, \ldots, \mathcal{U}_{m+1}\}$. Let $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_m$ and $\mathcal{V} = \mathcal{U}_{m+1}$ so that $M = \mathcal{U} \cup \mathcal{V}$. Observe that $\mathcal{U}$ is of finite type m, $\mathcal{V}$ is contractible, and

$$\mathcal{U} \cap \mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_m\,, \quad \text{where } \mathcal{V}_j = \mathcal{U}_j \cap \mathcal{U}_{m+1},$$

is of finite type m. Therefore, by induction hypothesis, all de Rham spaces of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{U} \cap \mathcal{V}$ are finite-dimensional. We know that $H^0(M)$ is finite-dimensional (why?) so fix $k \geq 1$. By Mayer-Vietoris we have an exact sequence

$$\cdots \to H^{k-1}(\mathcal{U} \cap \mathcal{V}) \xrightarrow{\delta} H^k(M) \xrightarrow{r} H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) \to \cdots\,. \tag{2.107}$$

Now the "dimension theorem" or "rank theorem" of linear algebra can be stated as follows: if $L : V \to W$ is a linear map between vector spaces V and W (not necessarily finite-dimensional) such that $\ker L$ and $\operatorname{Im} L$ are both finite-dimensional, then V is finite-dimensional and $\dim V = \dim \ker L + \dim \operatorname{Im} L$. (The "dimension theorem" usually assumes V and W are finite-dimensional, but you don't need to.) Thus, in view of (2.107), to prove that $H^k(M)$ is finite-dimensional we just have to prove that $\ker r$ and $\operatorname{Im} r$ are finite-dimensional. However, since $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{U} \cap \mathcal{V}$ have finite-dimensional cohomologies,

$$\dim(\ker r) = \dim(\operatorname{Im} \delta) \leq \dim H^{k-1}(\mathcal{U} \cap \mathcal{V}) < \infty$$

and

$$\dim(\operatorname{Im} r) \leq \dim \Big(H^k(\mathcal{U}) \oplus H^k(\mathcal{V})\Big) < \infty.$$

□

One of my favorite applications of de Rham cohomology is to prove the fundamental theorem of algebra; see Problem 6.

EXERCISES 2.9.

1. (**Deformation retract**) Let M and N be manifolds and suppose that $N \subseteq M$ and the inclusion map $\iota : N \to M$ is smooth. A **deformation retract** of M into N is a smooth map $r : M \to N$ such that

$$r \circ \iota = \mathrm{Id}_N \quad \text{and} \quad \iota \circ r \simeq \mathrm{Id}_M.$$

Prove that M and N have the same de Rham cohomology spaces.

2. Here are some exact sequence problems.
   (i) Prove that if $f : V \to W$ is any linear map between vector spaces, then the following sequence is exact:

$$0 \to \ker f \to V \xrightarrow{f} W \to W/\operatorname{Im} f \to 0,$$

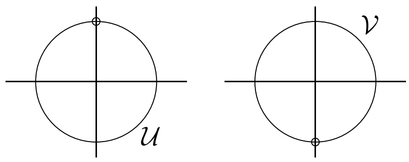


FIGURE 2.26. $\mathcal{U} = \mathbb{S}^1 \setminus \{(0,1)\}$ and $\mathcal{V} = \mathbb{S}^1 \setminus \{(0,-1)\}$.

where the map $\ker f \to V$ is inclusion and $W \to W/\operatorname{Im}(f)$ is projection $w \mapsto [w] \in W/\operatorname{Im} f$.

(ii) If $V_1, \ldots, V_k$ are finite-dimensional vector spaces that form an exact sequence

$$0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-2}} V_{k-1} \xrightarrow{f_{k-1}} V_k \to 0,$$

prove that the alternating sum of dimensions of the vector spaces vanish:

$$\sum_{i=1}^{k} (-1)^{i-1} \dim V_i = \dim V_1 - \dim V_2 + \cdots + (-1)^{k-1} \dim V_k = 0.$$

Apply this formula to (2.106) to immediately deduce that $\dim H^1(\mathbb{P}) = 1$.

3. In this problem we find the cohomology spaces of multi-punctured planes.
   (i) Let $a, b \in \mathbb{R}^2$ be distinct and compute $H^1(\mathbb{R}^2 \setminus \{a,b\})$. Suggestion: Let $\mathcal{U} = \mathbb{R}^2 \setminus \{a\}$ and $\mathcal{V} = \mathbb{R}^2 \setminus \{b\}$ and observe that $\mathbb{R}^2 = \mathcal{U} \cup \mathcal{V}$ and $\mathbb{R}^2 \setminus \{a,b\} = \mathcal{U} \cap \mathcal{V}$. Noting that $\mathcal{U}$ and $\mathcal{V}$ are diffeomorphic to the punctured plane, apply Mayer-Vietoris to $\mathbb{R}^2 = \mathcal{U} \cup \mathcal{V}$.
   (ii) Now with this practice let's make things a little more complicated. Let $a_1, \ldots, a_n \in \mathbb{R}^2$ be distinct and prove that

$$H^k(\mathbb{R}^2 \setminus \{a_1, \ldots, a_n\}) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^n & k = 1 \\ 0 & k > 1. \end{cases}$$

   Suggestion: Prove this by induction on n. Let $\mathcal{U} = \mathbb{R}^2 \setminus \{a_n\}$ and $\mathcal{V} = \mathbb{R}^2 \setminus \{a_1, \ldots, a_{n-1}\}$, then observe that $\mathbb{R}^2 = \mathcal{U} \cup \mathcal{V}$ and $\mathbb{R}^2 \setminus \{a_1, \ldots, a_n\} = \mathcal{U} \cap \mathcal{V}$. Note that $H^k(\mathcal{V})$ is known by induction hypothesis.
4. Here are some more Mayer-Vietoris problems. Try your hand at a couple.
   (i) Using the sets $\mathcal{U}$ and $\mathcal{V}$ in Figure 2.26, compute the cohomology spaces of $\mathbb{S}^1$.
   (ii) Compute de Rham cohomology spaces of the 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$.
   (iii) Here's a nice (but challenging) project: Try to prove that the n-torus $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (where there are n factors of $\mathbb{S}^1$) satisfies

$$\dim H^k(\mathbb{T}^n) = \binom{n}{k}, \quad 0 \le k \le n.$$

   (iv) Using the Mayer-Vietoris sequence, compute the de Rham cohomologies of Examples (iii) or (iv) in Problem 1 of Exercises 2.8.
5. For $n \ge 2$, show that

$$H^k(\mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{R} & k = 0, n-1 \\ 0 & k \ne 0, n-1. \end{cases}$$

   Using this result prove that $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^m$ if and only if $n = m$. Conclude that a manifold cannot be both an n-dimensional manifold and an m-dimensional manifold for $n \ne m$.
6. (**The fundamental theorem of algebra**) In this problem we give a de Rham theorem proof of the fundamental theorem of algebra.

(i) For each positive integer n, define $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ by $p_n(z) := z^n$ where elements of $\mathbb{S}^1$ are thought of as complex numbers of length one. We can also define p_n as follows using de Moivre's formula:

$$p_n(\,(\cos\theta, \sin\theta)\,) := (\,\cos(n\theta), \sin(n\theta)\,)\,, \quad \theta \in \mathbb{R}.$$

Using the definition of push-forward, prove that

$$(p_n)_*(\partial_\theta) = n\,\partial_\theta$$

where ∂_θ is the vector field specified in Problem 2 of Exercises 2.4.

(ii) Using (i), prove that

$$p_n^*(d\theta) = n\,d\theta, \tag{2.108}$$

where $d\theta$ is by definition the dual of ∂_θ. (Alternatively, if you want to take the definition of $d\theta$ as in Example 2.40 in Section 2.8, you can go ahead and prove (2.108) from that definition.) Using (2.108), prove that as a map on cohomology, $p_n^* : H^1(\mathbb{S}^1) \to H^1(\mathbb{S}^1)$ is just multiplication by n.

(iii) Let $p(z) = z^n + a_{n-1}\,z^{n-1} + \cdots + a_1\,z + a_0$ be a polynomial with complex coefficients with $n \geq 2$; we shall prove that $p(z)$ must have a zero. We assume $n \geq 2$ because $n = 1$ is obvious. Prove that there is an $r > 0$ such that for all $z \in \mathbb{C}$ with $|z| \leq r$, we have

$$|q(z)| < r^n\,, \quad \text{where} \quad q(z) := a_{n-1}\,z^{n-1} + \cdots + a_1\,z + a_0.$$

We shall analyze $f^* : H^1(\mathbb{S}^1) \to H^1(\mathbb{S}^1)$ where

$$f : \mathbb{S}^1 \to \mathbb{S}^1\,, \quad \text{is defined by} \quad f(z) := \frac{p(rz)}{|p(rz)|} \quad \text{for all } z \in \mathbb{S}^1.$$

(iv) Let $g : \mathbb{R} \to \mathbb{R}$ be the function in Figure 2.1 of Section 2.1 with $\varepsilon = 1$ in that Figure; this function g satisfies $0 \leq g \leq 1$, g is nondecreasing, $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t \geq 1$. Consider the homotopy $H_1 : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ defined by

$$H_1(z,t) := \frac{(rz)^n + g(t)q(rz)}{|(rz)^n + g(t)q(rz)|}.$$

Prove that the denominator is never zero so this function is well-defined. Using this homotopy, prove that $f^* = n\,\mathrm{Id}$ as a map on $H^1(\mathbb{S}^1)$.

(v) To prove that $p(z)$ must have a root, that is, there is a complex number $z \in \mathbb{C}$ such that $p(z) = 0$, assume by way of contradiction that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Consider the homotopy $H_2 : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ defined by

$$H_2(z,t) := \frac{p(trz)}{|p(trz)|}.$$

Prove that the denominator is never zero so this function is well-defined. Using this homotopy, prove that $f^* = \mathrm{Id}$ as a map on $H^1(\mathbb{S}^1)$. Since $n \geq 2$, this contraction proves the fundamental theorem of algebra.