RESEARCH STATEMENT

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My field of research is geometric group theory. The fundamental approach of geometric group theory is to extract algebraic information about the behavior of groups from geometric or topological properties of spaces on which they act. Geometric group theory has revitalized the study of groups and is a flourishing area of current research. It combines the techniques and tools of a number of areas, including classical group theory, algebraic and geometric topology, Riemannian geometry, and combinatorics. Applications of viewing algebraic objects from a geometric perspective have had far-reaching implications, into fields such as low-dimensional topology, Lie theory, hyperbolic geometry, logic and model theory, and computational complexity theory.

My research deals with topics related to classical Artin braid groups, including (right-angled) Artin groups, configuration spaces, braid groups on graphs, the mapping class group, and some applications to robotics.

To date, much of my research has been on braid groups on graphs, which are described in Section 1. Graph braid groups, introduced (in their current generality) by Abrams and Ghrist [1, 21], are fundamental groups of certain configuration spaces, and can simulate the motion of robots on a factory floor. They have connections to general Artin groups, especially to right-angled Artin groups. My research has involved tools such as Forman's discrete Morse theory [18] and reinterpreting some version of differential forms on discrete spaces to characterize group cohomology. My results include computing presentations for graph braid groups, analyzing the ring structure for cohomology of tree braid groups, embedding right-angled Artin groups into graph braid groups, and solving the isomorphism problem for a class of tree braid groups.

I am currently working on three main topics. In my proposed research, I plan to consider some remaining questions about graph braid groups. Secondly, in conversations about graph braid groups with engineers and roboticists, I have also become intrigued with other mathematical aspects of robotic motion. Thirdly, I am working with colleague Moon Duchin on bordifications of non-CAT(0) spaces - especially Cayley graphs of finitely generated groups. In the process, we are considering the interrelations between many different curvature conditions.

1. Background

1.1. Configuration spaces.

Configuration spaces provide a theoretical framework in which to analyze motions of objects. Consider a (possibly abstract) physical system $\mathcal{P}$, like a pendulum, or racecars moving about a racetrack. A *configuration* is one possible state $\mathcal{P}$ can be in, subject to the constraints of the system. Let $\mathcal{C}(\mathcal{P})$ denote the set of all possible configurations of the system $\mathcal{P}$. The system prescribes ways (often given by continuous motions or the objects involved) to transition from one configuration to another, inducing a topology on $\mathcal{C}(\mathcal{P})$. The space $\mathcal{C}(\mathcal{P})$ is the *configuration space* of the system $\mathcal{P}$. For instance, the configuration space of a pendulum is homeomorphic to a circle, where each point on the circle prescribes the angle between the pendulum and 'straight down', specifying a configuration.

Configuration spaces are versatile and frequently used tools in physics, mechanical engineering, robotics, topology, and many other disciplines. Hence, their study has many applications. For instance, configuration spaces have connections with microfluidics [20], robot sensor networks [1], mechanical arm linkages [24], and robot motion [21].

We will discuss in some detail about a particular kind of configuration space, which we define here. Let X be an arbitrary topological space, and let $\mathcal{P}$ denote the system of n distinct labelled points moving about on X. The configuration space $\mathcal{C}(\mathcal{P})$ is also denoted $\mathcal{C}^n X$, called the *labelled configuration space of n strands on X* (the choice of the term strand will become clear shortly). If instead the points are indistinguishable, the resulting configuration space is denoted $U\mathcal{C}^n X$, called the *unlabelled configuration space of n strands on X*. More topologically, $\mathcal{C}^n X$ is the open subset of the direct product $\Pi^n X$ obtained by removing the *fat* *diagonal*, $Diag := \{(x_1, \ldots, x_n) | x_i = x_j \text{ for some } i \neq j\}$. The fat diagonal represents two or more points colliding. The space $U\mathcal{C}^n X$ is the quotient of $\mathcal{C}^n X$ by the action of the symmetric group permuting the product factors.

1.2. **Braid groups.** Let X be a topological space, and fix a positive integer n. The fundamental group of $\mathcal{C}^n X$ is called the *pure braid group* of n strands on X, denoted $PB_n X := \pi_1 \mathcal{C}^n X$. The fundamental group of $U\mathcal{C}^n X$ is called the *braid group* of n strands on X, denoted $B_n X := \pi_1 U\mathcal{C}^n X$. Elements of $B_n X$ (respectively, $PB_n X$) are called *braids* (respectively, *pure braids*) on X.

Braids get their name from the following visual interpretation. A (pure) braid is a homotopy equivalence class of loops in the associated configuration space. Each point on such a loop is a configuration of n distinct points on X, and the loop corresponds to those n points moving around continuously in X without colliding. We can introduce a time parameterization to the paths these n points take - say from 0 to 1. We may then visualize the loop as n disjoint paths in $X \times [0, 1]$, beginning at some point in $X \cong X \times 0$ and ending at the same point in $X \cong X \times 1$. When X is the unit disk, these pictures look literally like braids.

The case when X is a disk is the classical example of a braid group. In this case, we call $B_n X$ *the* (classical, or Artin's) braid group on n strands, denoted B_n. Classical braid groups, originally studied by Artin [2], are fundamental objects in topology. They have a large body of literature concerning them, and have applications to numerous areas of mathematics, including knot theory, low-dimensional topology, Riemannian geometry, and mathematical biology. Braid groups are examples of a mapping class group, of fundamental importance in their own right, and also well studied. For a reference on braids, see [5].

As mentioned above, configuration spaces may be used to model the movements of robots about a factory floor [8, 21, 12, 13]. Consider n identical robots running around, performing tasks, moving boxes, etc. Such robots have certain constraints placed on their movement: they should not run into each other, and they must avoid obstacles and walls. Often in real world settings, such robots are restricted further, constrained to moving along tracks in the floor or ceiling. Such systems are currently in use in industry. Letting Γ be a graph representing the tracks of some factory floor or ceiling, $B_n\Gamma$ models the possible motions of n robots in the factory. When $X = \Gamma$ is a graph, the associated braid group is called a *graph braid group*.

Let Γ be a given finite graph. As Γ is a graph, it is also a 1-dimensional CW-complex, so that there is a product CW-complex structure on $\Pi^n\Gamma$. Let $Diag'$ be the set of all open cells of $\Pi^n\Gamma$ whose closure intersects $Diag$. The *labelled discretized configuration space* $\mathcal{D}^n\Gamma$ of n strands on a finite graph Γ is the CW-complex $\Pi^n\Gamma - Diag'$. The *unlabelled discretized configuration space* $U\mathcal{D}^n\Gamma$ of n strands on Γ is the quotient of $\mathcal{D}^n\Gamma$ given by permuting the factors of $\Pi^n\Gamma$ via the action of S_n.

Under most circumstances, the unlabelled (respectively, labelled) configuration space of Γ is homotopy equivalent to $U\mathcal{D}^n\Gamma$ (respectively, $\mathcal{D}^n\Gamma$) (for $n = 2$, see [29]; for general n, see [1]). For our purposes, we may safely assume this is always the case.

We mention here two important properties of graph braid groups proven by other authors. Firstly, Ghrist showed in [21] that the spaces $\mathcal{C}^n\Gamma$ are $K(PB_n\Gamma, 1)$s, of bounded homological dimension. Secondly, in [1], Abrams proved that $\mathcal{D}^n\Gamma$ is a locally CAT(0) cubical complex, so that graph braid groups have solvable word and conjugacy problems [6].

1.3. **Right-angled Artin groups.** Call a graph is *simple* if it has no closed non-backtracking edge paths of less than 3 edges. Let Δ be a finite simple graph. We examine a group $A = G(\Delta)$, called the *right-angled Artin group* associated to Δ, defined with the following presentation: for each vertex a_i of Δ, there exists a corresponding generator for A, and two generators a_i and a_j commute (for $i \neq j$) if and only if the vertices a_i and a_j are connected by an edge in Δ.

Right-angled Artin groups are examples of the more general class of groups called Artin groups, and are related to Coxeter groups. The classical braid groups are themselves Artin groups (though they are not right-angled). Like braid groups, right-angled Artin groups are well-studied [9]. They are particularly useful as sources of subgroups with complicated and interesting homotopy and homology finiteness behavior [28, 3].

1.4. **Discrete Morse theory.** We use Forman's *discrete Morse theory* [18] (see also [7, 3]) in order to simplify any complex $U\mathcal{C}^n\Gamma$ within its homotopy type. Given a CW-complex X and a Morse matching on X (equivalent to a discrete version of a Morse flow), discrete Morse theory classifies the cells of X as either critical or not. Critical cells correspond to cells in X which are "critical" to homotopy, homology, and other homotopy invariants. The collection of critical cells in a complex is usually much smaller than the original complex. Morse theory thus provides an efficient way for computing these homotopy invariants.

It is worth noting that Bestvina-Brady Morse theory [3] is at least a priori insufficient in the setting of discretized configuration spaces of graphs: in this context, there is no natural "height" function in the sense of [3].

2. PREVIOUS RESEARCH

2.1. Discrete Morse theory and presentations.

An important tool in studying groups is their presentations. For classical braid groups, presentations have played a pivotal role in the development of the subject. The new Garside presentations of the classical braid groups, due to Birman, Ko, and Lee [4], for instance, gave a new approach to the word and conjugacy problems and spurred on intensive research in braid theory.

Efficient presentations for graph braid groups have been elusive. Before Dan Farley and I applied Forman's *discrete Morse theory* [18] to graph braid groups in [16], presentations for graph braid groups were known only in isolated cases. In [16], Farley and I developed a Morse matching for $U\mathcal{D}^n\Gamma$. Using the resulting classification of critical cells, we show among other consequences a general method for obtaining presentations for any graph braid group, called *Morse presentations*, where generators correspond to critical 1-cells and relators correspond to boundary words of critical 2-cells. The generators in these presentations are particularly beautiful, and may be visualized geometrically, as in Figure 1(a). Morse presentations for trees are in a definable sense minimal [14]. Farley and I are making some progress in minimizing presentations for general graphs in [15].

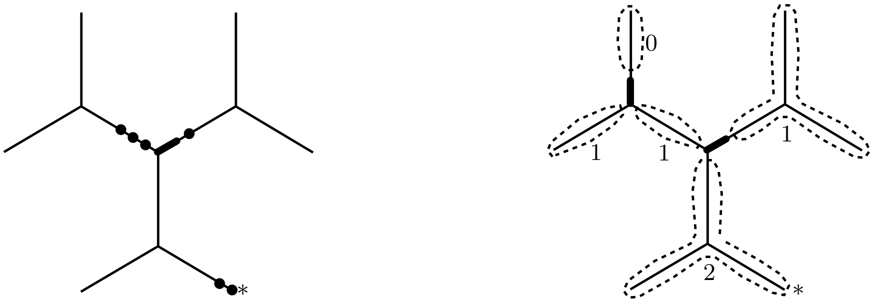

FIGURE 1. Figure 1(a) shows one possible critical 1-cell on the given graph. Each thick vertex or edge represents a strand. Figure 1(b) shows a 'cloud diagram' representing a cohomology class. Taking the dual of the critical 1-cell in Figure 1(a), there exists a distinct second 1-cohomology class such that cup product of the two is the 2-cohomology class of Figure 1(b).

2.2. Cohomology and the isomorphism problem.

Discrete Morse theory applied to the configuration space for a graph braid group is useful for more than just computing presentations. Since discretized configuration spaces of graphs are $K(G,1)$s, discrete Morse theory can be used to compute homotopy invariants like homology and cohomology. For instance, using the Morse matching developed in [16], in [14], Farley showed that $H_i(U\mathcal{C}^nT;\mathbb{Z})$ (equivalently, $H_i(B_nT;\mathbb{Z})$, since $U\mathcal{C}^n\Gamma$ is aspherical for any graph Γ [1, 21]) is a free abelian group of computable rank.

In [17], Farley and I compute $\mathbb{Z}/2\mathbb{Z}$-cohomology for braid groups of trees. We have not only been able to compute the useful tool of cohomology groups for these tree braid groups, but we have also used discrete Morse theory to describe the actual cohomology ring structure by describing the behavior of cup products (see Figure 1(b)).

Understanding the ring structure has already proven to be a powerful tool in analyzing the groups themselves. Knowing the cup product structure for tree braid groups allowed us to completely answer which tree braid groups are right-angled Artin groups (see next section). Also using the ring structure, in [26] I prove:

Theorem 2.1. *Let T and T' be two finite trees, and let $n = 4$ or 5. Then T and T' are homeomorphic if and only if B_nT and B_nT' are isomorphic.*

That the tree braid groups may determine their graphs is an isolated phenomenon; even for two dimensional spaces, for any n there exist infinitely many connected complexes X for which $B_nX \cong B_n$, the n-strand braid group.

To prove this, I first show that, for any finite tree T and $n = 4$ or 5, $H^*(B_nT; \mathbb{Z}/2\mathbb{Z})$ is an *exterior face algebra* (I conjecture that this does not hold for most trees with $n \geq 6$). An exterior face algebra is the quotient of an exterior algebra by certain ideals determined by the faces of a simplicial complex Δ. I prove that for a four- or five-strand tree braid group, the simplicial complex Δ defining the exterior face algebra structure of $H^*(B_nT; \mathbb{Z}/2\mathbb{Z})$ is unique and 1-dimensional.

I use the bijection between finite trees up to homeomorphism and n strand tree braid groups up to isomorphism to find a solution to the isomorphism problem for n-strand tree braid groups, $n = 4$ or 5. The isomorphism problem is one of the fundamental problems in geometric group theory. It asks, given two group presentations of some particular type of group, is it possible to determine algorithmically whether the two corresponding groups are isomorphic:

Theorem 2.2. *Let G and G' be two groups given by finite presentations, and assume that $G \cong B_nT$ and $G' \cong B_nT'$ for some positive integer n and finite simple trees T and T'. If either:*

- *$n = 4$ or 5 or*
- *at least one of G or G' is free,*

then there exists an algorithm which decides whether G and G' are isomorphic. The trees T and T' need not be specified. If one of T and T' has at least 3 vertices of degree at least three, then n need not be specified.

2.3. **Right-angled Artin groups and surface subgroups.** The relationship between graph braid groups and right-angled Artin groups has often been speculated upon and explored. In 1998, Ghrist [21] conjectured that the (pure) braid group of any graph is a right-angled Artin group. Although this conjecture is true in some cases [10], it is not true in general [22, 1]. Following the work of Glover, Abrams revised Ghrist's conjecture to state:

Conjecture 1. [1, 21] *The braid group of any planar graph is a right-angled Artin group.*

Farley and I answer this conjecture in the context of tree braid groups in [17]. A *linear tree* is a tree in which all vertices of degree at least three lie on an embedded line segment. We use the ring structure on cohomology to find a "poisonous subring" in the cohomology ring of all tree braid groups of nonlinear trees (with $n \geq 4$ strands) which prevents the groups from being right-angled Artin:

Theorem 2.3. [17] *Let T be a finite tree. The tree braid group B_nT is a right-angled Artin group if and only if T is a linear tree or $n < 4$.*

In general, it appears that graph braid groups are usually not right-angled Artin groups. Mautner [25] has found examples of graph braid groups on trees and on non-simply-connected planar graphs which are not right-angled Artin.

A related line of inquiry asks whether there exist embeddings between right-angled Artin groups and graph braid groups. According to [11], for any finite graph Γ and any n, there exists a graph Δ such that $B_n\Gamma$ embeds in $G(\Delta)$. Thus, graph braid groups are linear, bi-orderable, and residually finite. I have proven the other direction:

Theorem 2.4. [27] *For every finite graph Δ there exists a graph Γ such that $G(\Delta)$ embeds into $B_n\Gamma$, where n is the chromatic number of Δ.*

The proof of this theorem introduces the construction of a "graph halo", that is inspired by physical considerations. Given an arbitrary Δ, I define a homomorphism $f\colon G(\Delta) \to B_n\Gamma$, where Γ is a graph which contains a simple loop for every vertex of Δ, with two such simple loops touching if and only if the corresponding elements of $G(\Delta)$ do not commute. Let $h\colon B_n\Gamma \to G(\Delta')$ denote the Crisp-Wiest embedding. To prove that f is injective it suffices to show that the composition $h \circ f\colon G(\Delta) \to G(\Delta')$ is injective. This task is accomplished by using the HNN-structure of right-angled Artin groups and iterated use of Britton's lemma.

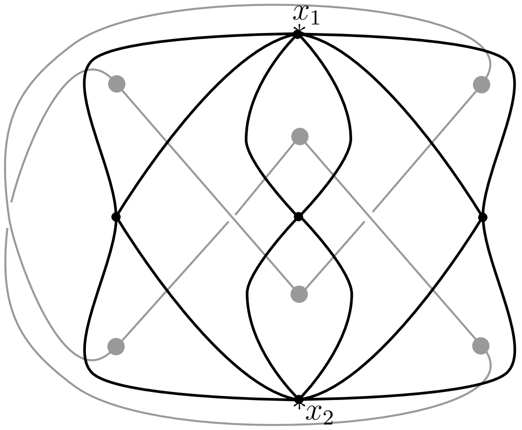

FIGURE 2. This figure shows a possible C_6-halo $\Gamma = \Gamma(C_6)$, with the graph C_6 subimposed in gray. Theorem 2.4 shows that the braid group $B_2\Gamma$ contains a hyperbolic surface subgroup, as $G(C_6)$ contains one.

When I apply this theorem to a specific graph in [27] (see Figure 2), I prove:

Theorem 2.5. *There exists a planar graph braid group which contains as a subgroup the fundamental group of a closed hyperbolic surface, with only $n = 2$ strands.*

There were previously no known examples of hyperbolic surface subgroups in planar graph braid groups.

3. FUTURE DIRECTIONS

3.1. **Robotics.** The question of studying graph braid groups has its origins in mathematical robotics. As mentioned, a graph braid group can model the motions of automated guided vehicles (that is, robots) in factories. I am researching two questions of a similar vein - that is, mathematical aspects of robot motion - with immediate applied implementations possible.

The first question I am addressing is for online mobile robot navigation. Consider a spherical 2-dimensional robot moving in a 2-dimensional environment. An example of this is a Roomba vacuum. Let S denote the starting point of the robot, and let T denote the target position of the robot. We wish to program the robot to move from S to T with no a priori knowledge of the environment. Gabriely and Rimon [19] have developed an algorithm, called CBUG, which solves this problem in time which is quadratic with respect to the shortest path between S and T. They prove that quadratic *competitiveness* is optimal for this problem.

Three immediate questions to ask about generalizing CBUG are: is it possible to remove the dimension restriction? What about more general (nonplanar) spaces? And, what about robots of varying shapes? I am working an algorithm which generalizes the CBUG algorithm to online navigation on an arbitrary space with global coordinates, regardless of dimension. I believe my methods will work for arbitrary ellipsoid-shaped robots. One key observation necessary for my generalization is that the robot does not need to know small scale aspects of the contours of obstacles. A further direction with this algorithm is to try to develop an algorithm which 'usually' runs in linear time with respect to the shortest path between S and T. In the worst case situation, any online algorithm must run in time proportional to the dimension of the space, but can one improve this bound in general?

A further elaboration of this question is along the lines of sensor networks. What if the space in question has no global coordinates? In place of coordinates, our robot is able to leave behind beacons which broadcast over a short range. Nearby beacons can sense each other, as can the robot, but there is no direct link from beacons far away to the robot. In this kind of decentralized network of sensors, is it still possible to find a solution to online robot navigation? I believe my proposed algorithm in the global coordinate case may be modified to this coordinate-free scenario.

Finally, one further elaboration I plan to explore is to allow the target point to move.

A related online navigation question asks for efficient algorithms for determining motions of mechanical arm linkages. An arm linkage is a series of rigid segments connected by joints, where each joint can move and attain configurations corresponding to a subset of a sphere. The space of configurations of an arm linkage is thus some subspace of the product of configuration spaces for each joint, and so is a subspace of a product of spheres. As a product of spheres can be given global coordinates, so can the configuration space of a mechanical arm linkage. To move a mechanical arm linkage from a given configuration to another is equivalent to asking for a continuous path in the space of configurations between the given configuration and the desired configuration - that is, between two points in the configuration space. Thus, the question of motion for mechanical arm linkages may be solved by applying my generalization of the CBUG algorithm. NASA has expressed interest in this particular application, and I plan to tailor my results to this situation.

3.2. **Graph braid groups.** Although I have diversified my interests and am thinking about problems in other areas, there are still some open questions for graph braid groups that I am considering. Farley and I have developed a decent understanding of the behavior of tree braid groups, but not much is understood for braid groups on more general graphs.

For instance, the theorems in [16] give a number of presentations for an arbitrary graph braid group. In the case of tree braid groups, those presentations are optimal [14]. For general graph braid groups, however, the presentations are usually full of extraneous generators and relators, and are typically overly complicated. Farley and I are making some progress in [15] toward these more general presentations, especially in the two-strand case. Similarly, I want to better understand the homology and cohomology of general graph braid groups. For tree braid groups, we have the advantage that when looking at homology all of the boundary maps are 0, but this does not hold for general graph braid groups. A related question is to characterize the cohomological dimension of graph braid groups.

An interesting application of graph braid groups may lie in Henry Glover's Kuratowski Covering Conjecture: A finite graph Γ fails to embed in a non-orientable surface $\tilde{S}_g$ if and only if there exists $g+1$ Kuratowski subgraphs such that each pair fails to embed in $\mathbb{R}P^2 = \tilde{S}_1$ and each triple fails to embed in $\tilde{S}_2$, the 2-torus.

This problem relates to and tries to generalize the classical result of Kuratowski that a graph is planar if and only if it does not contain either of K_5 or $K_{3,3}$ - so called Kuratowski graphs - as a subgraph. Glover has checked his conjecture for a large number of small graphs, but believes the key to answering the conjecture lies in two-strand graph braid groups. Glover believes that a graph Γ embeds in $\tilde{S}_g$ if and only if, when Γ is 2-connected, there exists an equivariant map ϕ from $C^2\Gamma$ to $C^2\tilde{S}_g$, and that this result implies the Kuratowski Covering Conjecture. My research circumstantially supports the conjecture, and reveals a division between two-strand graph braid groups on planar and nonplanar graphs, not only in terms of hyperbolic surface subgroups but also at the level of presentations. Porting these observations to homology and cohomology may yield the desired solution to the Kuratowski Covering Conjecture.

3.3. **Curvature conditions and boundaries.** Let X be a metric space. In [23], Gromov defined the *m-curvature condition space* $K_m(X)$ of X to be the subspace of $m \times m$ symmetric matrices whose entries correspond to distances between configurations of m points in X. For example, if $m = 3$, then an element of $K_3(X)$ is determined by 3 values, which correspond to the pairwise distances between three points in X. Let Tri_m denote the space of all $m \times m$ symmetric matrices whose entries satisfy the triangle inequality, when thought of as distances between m points. Then $K_m(X) \subseteq Tri_m$ for all m. A *(proper) m-curvature condition* is a (proper) subspace K' of Tri_m, and we say X *satisfies* the m-curvature condition K' if $K_m(X) \subseteq K'$. Typically, m-curvature conditions are specified by inequalities on distances between m-tuples of points.

For $m < 4$, curvature conditions are relatively well understood: a result of M. Kapovich [24] shows that a connected space which is not quasiisometric to a subset of $\mathbb{R}$ cannot satisfy any proper 3-curvature condition. For $m > 4$, little is known. But for $m = 4$, there is much to study. Examples of proper 4-curvature conditions include Gromov's δ-hyperbolicity and the $CAT(\kappa)$ curvature conditions. Moon Duchin and I are examining other interesting 4-curvature conditions, and their interrelations and implications. We have four potentially interesting 4-curvature conditions, distinct from δ-hyperbolicity and the $CAT(\kappa)$ conditions, many of which have implications for coarse geometric properties of the space.

Related to our study of 4-curvature conditions is our examination of boundaries for non-CAT(0) spaces. In CAT(0) spaces, the horofunction boundary and the visual boundary (geodesic rays up to asymptotic equivalence, topologized by uniform convergence on compact sets) always coincide. Being CAT(0) is not a necessary condition for this property, though, nor for the existence of a well-defined map from the horofunction boundary to the visual boundary. For instance, Duchin and I have proven that there is a well-defined map from the horofunction boundary of (the Cayley graph of) a right-angled Artin group (with respect to the standard generating set) to the visual boundary: every horofunction arises from a geodesic ray. Our goal is to push these results to describe the horofunction boundary of general finitely generated groups, and in particular the mapping class group as a topological space.

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