Extra Credit #1 Solution Set

February 7, 2006

(1.6#6) When solutions of sodium phosphate and barium nitrate are mixed, the result is barium phosphate (as a precipitate) and sodium nitrate. The unbalanced equation is

$$Na_3PO_4 + Ba(N0_3)_2 \to Ba_3(PO_4)_2 + NaNO_3.$$

Balance this equation.

Answer: We need to find $\vec{x}$ so that

$$x_1 Na_3PO_4 + x_2 Ba(NO_3)_2 \to x_3 Ba_3(PO_4)_2 + x_4 NaNO_3.$$

For each compound, we construct a vector which lists the number of atoms of sodium, phosphorus, oxygen, barium, and nitrogen, in that order. We have:

$$Na_3PO_4 \leftrightarrow \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix}, Ba(NO_3)_2 \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 6 \\ 1 \\ 2 \end{bmatrix}, Ba_3(PO_4)_2 \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 8 \\ 1 \\ 0 \end{bmatrix}, NaNO_3 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

To balance the equation, we need:

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 6 \\ 1 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 8 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

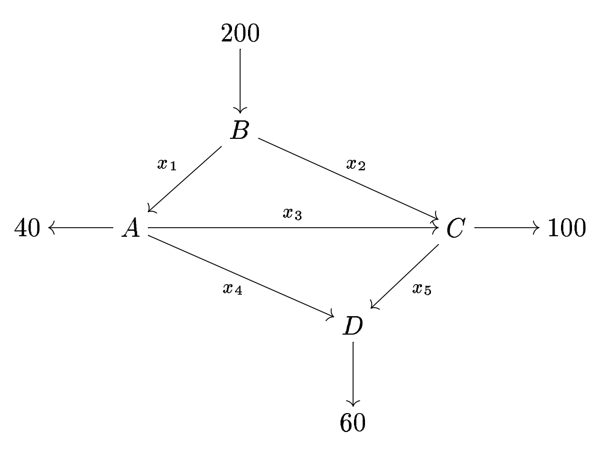
This is a vector equation. To find solutions, we row reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -1 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right] \longrightarrow \ldots \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This means the general solution is $x_1 = x_4/3$, $x_2 = x_4/2$, $x_3 = x_4/6$, and x_4 is free. Picking x_4 to be positive but small as possible and so that all four values are integers, we get $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, and $x_4 = 6$:

$$2Na_3PO_4 + 3Ba(NO_3)_2 \to Ba_3(PO_4)_2 + 6NaNO_3.$$

(1.6#12) Consider the following freeway network:



(a) Find the general traffic pattern in the freeway network shown.

Answer: There are four freeway intersections - A, B, C, D - each of which have an equal number of incoming and outgoing cars. The four corresponding linear equations are:

$$\begin{array}{rrcl} A: & x_1 & = & 40 + x_3 + x_4, \\ B: & 200 & = & x_1 + x_2, \\ C: & x_2 + x_3 & = & 100 + x_5, \\ D: & x_4 + x_5 & = & 60. \end{array}$$

Row reducing the augmented matrix corresponding to this system of equations yields:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ -1 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution to this system is thus:

$$\begin{cases} x_1 & = 100 + x_3 - x_5 \\ x_2 & = 100 - x_3 + x_5 \\ x_3 & \text{is free} \\ x_4 & = 60 - x_5 \\ x_5 & \text{is free} \end{cases}$$

(b) Describe the general traffic pattern when the road whose flow is x_4 is closed.

Answer: If the road whose flow is x_4 is closed, then $x_4 = 0$. Adding this linear equation to the system of linear equations arrived at in part (a), we see that:

$$\begin{cases} x_1 & = 40 + x_3 \\ x_2 & = 160 - x_3 \\ x_3 & \text{is free} \\ x_4 & = 0 \\ x_5 & = 60 \end{cases}$$

(c) When $x_4 = 0$, what is the minimum value of x_1?

Answer: Part (b) tells us that when $x_4 = 0$, $x_1 = 40 + x_3$. Since x_3 cannot be negative, x_1 must be at least 40.

(1.7#20) Determine by inspection whether or not the following vectors are linearly *independent*:

$$\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Answer: If $\vec{0}$ is ever in a set of vectors, that set of vectors is not linearly independent. These vectors are linearly dependent. For an example of a nontrivial combination of these vectors which sum to $\vec{0}$, take for instance:

$$\vec{0} = 0 \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} + -43256 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(1.7#41) Use as many columns of the following matrix A as possible to construct a matrix B with the property that the equation $B\vec{x} = \vec{0}$ has only the trivial solution. Solve $B\vec{x} = \vec{0}$ to verify your work.

$$A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$$

Answer: Row reducing A, we get that columns 1, 2, and 5 are pivot columns, while columns 3 and 4 are nonpivot columns. This means that there are two free variables, so we need to remove 2 columns from A to get a unique solution to $B\vec{x} = 0$. It also means that if we remove columns 3 and 4, the resulting matrix

$$B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

will have three pivot columns an no nonpivot columns. That there are no nonpivot columns means that $B\vec{x} = \vec{0}$ has a unique solution.

Note: Other matrices besides the B given here are possible solutions. If I had to guess (i.e. I have not checked the following statement), I would say that the following sets of three columns of A would form a possible matrix B: 1,2,5; 1,3,5; 1,4,5; 2,3,5; 2,4,5; 3,4,5.