Extra Credit #2 Solution Set

March 7, 2006

(3.2#26) Decide if the following set of vectors is linearly independent.

$$\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}.$$

Answer: Let $A = \begin{bmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & 3 \end{bmatrix}$. Computing $|A|$ by cofactor expansion down the second column and then across the last row, we have:

$$|A| = (-1)^{4+4}(-3) \begin{vmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{vmatrix} = (-3)\left((-1)^{1+3}(-6)\begin{vmatrix} 2 & -2 \\ -6 & -1 \end{vmatrix} + (-1)^{3+3}(3)\begin{vmatrix} 3 & 2 \\ 5 & -6 \end{vmatrix}\right) =$$
$$(-3)((-6)((-2) - (-2)(-6)) + 3(-18 - 10)) = (-3)(84 - 84) = 0.$$

This means that the columns of A are linearly dependent.

(3.2#28) True/False:

(a) If two row interchanges are made in succession, then the new determinant equals the old determinant.

Answer: True: each row interchange multiplies the determinant by (-1), and $(-1)(-1) = 1$.

(b) The determinant of A is the product of the diagonal entries in A.

Answer: False: this is only true for triangular matrices.

(c) If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.

Answer: False: see for instance Section 3.2 Problem # 26.

(d) $\det A^T = (-1)\det A$.

Answer: False: $\det A^T = \det A$.

(3.2#40) Let A and B be 4×4 matrices, with $\det A = -1$ and $\det B = 2$. Compute:

(a) $\det AB$

Answer: $\det AB = (\det A)(\det B) = (-1)(2) = -2$.

(b) $\det B^5$

Answer: $\det B^5 = (\det B)^5 = 2^5 = 32$.

(c) $\det 2A$

Answer: Multiplying A by 2 is the same as multiplying each row of A by 2. Since A has 4 rows and multiplying a row by a constant multiplies the determinant by the same constant, $\det 2A = 2^4 \det A = 2^4(-1) = -16$.

(d) $\det A^T A$

Answer: $\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = (-1)(-1) = 1$.

(e) $\det B^{-1}AB$

Answer: $\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = (\det B)^{-1}(\det A)(\det B) = (\det A) = -1.$

(3.3#2) Use Cramer's rule to compute the solutions of:

$$\begin{array}{rcl} 4x_1+ \quad x_2 & = & 6 \\ 5x_1+ \quad 2x_2 & = & 7 \end{array}.$$

Answer: The matrix equation $A\vec{x} = \vec{b}$ corresponding to this system of linear equations has $A = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$. Then $A_1(\vec{b}) = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}$ and $A_2(\vec{b}) = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$. Computing determinants, we have: $\det A = 3$, $\det A_1(\vec{b}) = 5$, and $\det A_2(\vec{b}) = -2$. By Cramer's rule,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}.$$

(3.3#11) Compute the adjugate of $A = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$, and then use Theorem 8 to give the inverse.

Answer: To compute the adjugate, we compute the cofactors:

$$\begin{array}{lll} C_{1,1} = +\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, & C_{1,2} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3, & C_{1,3} = +\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3, \\ C_{2,1} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = 1, & C_{2,2} = +\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1, & C_{2,3} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2, \\ C_{3,1} = +\begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0, & C_{3,2} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3, & C_{3,3} = +\begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6. \end{array}$$

Thus,

$$adj(A) = [C_{i,j}]^T = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}.$$

Since $\det A = 3C_{2,1} = 3$, by Theorem 8 we have:

$$A^{-1} = \frac{1}{\det A} adj(A) = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}.$$