Extra Credit #3 Solution Set

April 11, 2006

(5.2#10) Find the characteristic polynomial of $A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

Answer: We have:

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} \\ &= (-\lambda)^3 + 3*2*1 + 1*3*2 - (-\lambda)*2*2 - 3*3*(-\lambda) - 1*(-\lambda)*1 \\ &= -\lambda^3 + 14\lambda + 12. \end{aligned}$$

(5.2#19) Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicities, so that

$$det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda).$$

Explain why $\det A$ is the product of the n eigenvalues of A.

Answer: The formula above holds for any λ, so consider $\lambda = 0$. In this case, $\det(A) = \det(A - 0I) = \lambda_1\lambda_2\ldots\lambda_n$.

(5.2#27) Let $A = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

(a) Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors of A.

Answer: We have that $A\vec{v}_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = 1\vec{v}_1$, $A\vec{v}_2 = \begin{bmatrix} .5 \\ -1.5 \\ 1 \end{bmatrix} = .5\vec{v}_2$, and $A\vec{v}_3 = \begin{bmatrix} -.2 \\ 0 \\ .2 \end{bmatrix} = .2\vec{v}_3$.

(b) Let $\vec{x}_0$ be any vector in $\mathbb{R}^3$ with nonnegative entries whose sum is 1. Explain why there are constants c_1, c_2, c_3 such that $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. Compute $\vec{w}^T\vec{x}_0$, and deduce that $c_1 = 1$.

Answer: By part (a), the three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are eigenvectors for A which correspond to distinct eigenvalues. Since the eigenvalues are distinct, by Theorem 2, the eigenvectors are linearly independent and form an eigenbasis for $\mathbb{R}^3$. Since they form a basis, any vector in $\mathbb{R}^3$ - in particular, $\vec{x}_0$ - is a linear combination of the basis vectors.

Since each entry of $\vec{w}$ is 1, $\vec{w}^T\vec{x}_0$ is the sum of the entries of $\vec{x}_0$, which is 1. But then $1 = \vec{w}^T\vec{x}_0 = c_1\vec{w}^T\vec{v}_1 + c_2\vec{w}^T\vec{v}_2 + c_3\vec{w}^T\vec{v}_3$. Since $\vec{w}^T\vec{v}_1 = 1$, $\vec{w}^T\vec{v}_2 = 0$, and $\vec{w}^T\vec{v}_3 = 0$, we have $1 = c_1 + 0 + 0$.

(c) For $k = 1, 2, \ldots$, define $\vec{x}_k = A^K\vec{x}_0$, with $\vec{x}_0$ as in part (b). Show that $\vec{x}_k \to \vec{v}_1$ as k increases.

Answer: We know that $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, so We know that

$$\vec{x}_k = c_1A^k\vec{v}_1 + c_2A^k\vec{v}_2 + c_3A^k\vec{v}_3 = c_1(1)^k\vec{v}_1 + c_2(.5)^k\vec{v}_2 + c_3(.2)^k\vec{v}_3 \to c_1\vec{v}_1.$$

(5.3#6) The following matrix A is factored into the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

Answer: By the Diagonalization Theorem, the diagonal entries of D are the eigenvalues of A, so the eigenvalues are 5, 5, 4. The eigenspace for the eigenvalue 5 has as a basis $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. The eigenspace for the eigenvalue 4 has as a basis $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

(5.3#21) (a) A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

Answer: False: D must be diagonal.

(b) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

Answer: True: a basis is linearly independent, so by Theorem 5, A is diagonalizable.

(c) A is diagonalizable if and only if A has n eigenvalues, counting multiplicities. **Answer:** False: a diagaonalizable matrix must have n linearly independent eigenvectors.

(d) If A is diagonalizable, then A is invertible.

Answer: False: there is no correlation between diagonalizability and invertibility.