Homework #1 Solution Set

January 26, 2006

(1.1#4) Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.

Answer: The point of intersection between these two lines is a common solution to both equations. Therefore, we solve the linear system composed of the two equations. This corresponds to row reducing the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -5 & 1 \\ 3 & -7 & 5 \end{array}\right].$$

We have:

$$\left[\begin{array}{cc|c} 1 & -5 & 1 \\ 3 & -7 & 5 \end{array}\right] \xrightarrow{R_2'=R_2-3R_1} \left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & 8 & 2 \end{array}\right] \xrightarrow{R_2'=R_2/8} \left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & 1 & 1/4 \end{array}\right] \xrightarrow{R_1'=R_1+5R_2} \left[\begin{array}{cc|c} 1 & 0 & 9/4 \\ 0 & 1 & 1/4 \end{array}\right],$$

so $x_1 = 9/4$ and $x_2 = 1/4$ is the point of intersection.

(1.1#17) Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

Answer: Yes, they have a common point of intersection. A point of intersection between these three lines is a common solution to the equations. Therefore, we solve the linear system composed of the three equations. This corresponds to row reducing the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right].$$

We have:

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right] \xrightarrow{R_2'=R_2-2R_1,R_3'=R_3+R_1} \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{array}\right] \xrightarrow{R_3'=R_3+R_2} \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right].$$

At this point, we know we can solve for x_2 (in particular, $x_2 = -5/7$) and using this value for x_2 we could solve for x_1 (in particular, $x_1 = -13/7$). Thus, a solution exists, so there is a point of intersection.

(1.2#4) Row reduce the following matrix to reduced echelon form. Name the pivot positions in the RREF and the pivot columns in the original matrix.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

Answer: We have:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \xrightarrow{R_2'=R_2-3R_1, R_3'=R_3-5R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$
$$\xrightarrow{R_3'=R_3-2R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$
$$\xrightarrow{R_3'=-R_3/10} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$
$$\xrightarrow{R_3'=-R_3/10} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_1'=R_1-7R_3, R_2'=R_2+12R_3} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_2'=-R_2/4} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_1'=R_1-3R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The pivot positions are: row 1, column 1; row 2, column 2; row 3, column 4. The pivot columns are the first, second and fourth columns.

(1.2#13) Find the general solution of the system whose augmented matrix is:

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Answer: We have:

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1'=R_1+R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R_1'=R_1+3R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the general solution is

$$\begin{cases} x_1 & = 3x_5 + 5 \\ x_2 & = 4x_5 + 1 \\ x_3 & \text{is free} \\ x_4 & = -9x_5 + 4 \\ x_5 & \text{is free} \end{cases}$$

(1.2#26) Supppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.

Answer: If the 3×3 coefficient matrix has a pivot position in each column, then the augmented matrix has a pivot position in the first three columns. Since there can only be as many pivots as rows, there are only 3 pivot positions in the augmented matrix. This means there is not a pivot position in the last column of the augmented matrix. Theorem 2 then tells us that the system is consistent and, since there are no free variables, the solution is unique.

(1.3#17) Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is $\vec{b}$ in the plane spanned by $\vec{a}_1$ and $\vec{a}_2$?

Answer: The vector b is in the plane spanned by $\vec{a}_1$ and $\vec{a}_2$ if b is a linear combination of $\vec{a}_1$ and $\vec{a}_2$. Thus, we want to solve:

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}.$$

This vector equation has the same solution set as the system of linear equations corresponding to the augmented matrix $[\vec{a}_1\vec{a}_2|\vec{b}]$, so we row reduce this matrix:

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{array}\right] \xrightarrow{R_2'=R_2-4R_1, R_3'=R_3+2R_1} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{array}\right]$$
$$\xrightarrow{R_3'=R_3-3R_2/5} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 0 & h+17 \end{array}\right].$$

The system corresponding to this matrix is consistent if and only if $h + 17 = 0$ - i.e. if and only if $h = -17$.

(1.3#25) Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, and let $W = Span\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

a Is $\vec{b} \in \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$? How many vectors are in $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Answer: No, $\vec{b}$ is not in $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. There are only three vectors in $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ - specifically, $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ - and $\vec{b}$ is not one of them.

b Is $\vec{b} \in W$? How many vectors are in W? **Answer:** Yes, $\vec{b} \in W$, because $\vec{b}$ is a linear comination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. We know $\vec{b}$ is a linear comination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ because:

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right] \xrightarrow{R_3'=R_3+2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right]$$
$$\xrightarrow{R_3'=R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

is consistent.

The set W is the set of all linear combinations of $\vec{a}_1, \vec{a}_1, \vec{a}_3$, and so contains infinitely many vectors.

c Show that $\vec{a}_1 \in W$.

Answer: We write $\vec{a}_1$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$:

$$\vec{a}_1 = 1\vec{a}_1 + 0\vec{a}_2 + 0\vec{a}_3.$$