Homework #2 Solution Set

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(1.4#13) Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and let $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of A?

Answer: If $\vec{u}$ were in the plane spanned by the columns of A, then $\vec{u}$ would be a linear combination of the columns of A. This is the case if and only if the matrix equation $A\vec{x} = \vec{u}$ is consistent. To see whether $A\vec{x} = \vec{u}$ is consistent, we row reduce the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{array}\right]$$
$$\xrightarrow{R_2'=R_2+2R_1, R_3'=R_3-3R_1} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{array}\right]$$
$$\xrightarrow{R_3'=R_3+R_2} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{array}\right].$$

This matrix is in row echelon form, so we know that the original augmented matrix does not have a pivot position in the last column. Thus, the system is consistent (by Theorem 2), and $\vec{u}$ is indeed in the plane spanned by the columns of A.

(1.4#18) Do the columns of the following matrix B span $\mathbb{R}^4$? Does the equation $B\vec{x} = \vec{y}$ have a solution for each $\vec{y}$ in $\mathbb{R}^4$?

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

Answer: Theorem 4 tells us that both of these conditions are equivalent - the one is true if and only if the other is true. Theorem 4 also tells us how to test whether or not these conditions hold: the answer to both questions is yes if and only if B has a pivot position in every row. Row reducing B, we get:

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \xrightarrow{R_3'=R_3-R_1, R_4'=R_4+2R_1} B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix}$$
$$\xrightarrow{R_3'=R_3+R_2, R_4'=R_4+2R_2} B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$
$$\xrightarrow{R_3 \leftrightarrow R_4} B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix B only has 3 pivot positions, and does not have a pivot in each row. Thus, the answer to both questions is 'No'.

(1.4#31) Let A be a 3×2 matrix. Explain why the equation $A\vec{x} = \vec{b}$ cannot be consistent for all $\vec{b} \in \mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

Answer: For any matrix A with more rows than columns, there can only be as many pivot positions as columns. As there are more rows than columns, the augmented matrix $[A|\vec{b}]$ is allowed to have more pivot columns than A. If the augmented matrix has more pivot columns, this means the last column of the augmented matrix is a pivot column and Theorem 2 tells us the system is not consistent.

(1.5#12) Describe all solutions of $A\vec{x} = 0$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer: We row reduce the augmented matrix. We have:

$$\left[\begin{array}{cccccc|c} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2'=R_2+8R_3} \left[\begin{array}{cccccc|c} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1'=R_1-2R_2} \left[\begin{array}{cccccc|c} 1 & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the general solution is

$$\begin{cases} x_1 & = -5x_2 - 8x_4 - x_5 \\ x_2 & \text{is free} \\ x_3 & = 7x_4 - 4x_5 \\ x_4 & \text{is free} \\ x_5 & \text{is free} \\ x_6 & = 0 \end{cases}$$

In parametric vector form, the solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5.$$

(1.5#22) find a parametric equation of the line M through $\vec{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$.

Answer:

The line M is parallel to the vector $\vec{q} - \vec{p} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$. This means M is parallel to the line $Span(\vec{q} - \vec{p}) = \{(\vec{q} - \vec{p})t | t \in \mathbb{R}\}$. In fact, M is this line shifted by one of the points on M. In particular, M is the line whose parametric vector equation is

$$\vec{p} + (\vec{q} - \vec{p})t = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ -7 \end{bmatrix} t.$$

Equivalently, M is the line whose parametric vector equation is

$$\vec{q} + (\vec{q} - \vec{p})t = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 6 \\ -7 \end{bmatrix} t.$$