Homework #4 Solution Set

February 23, 2006

(2.1#16)

(a) If A and B are 3×3 and $B = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]$, then $AB = [A\vec{b}_1 + A\vec{b}_2 + A\vec{b}_3]$.
Answer: False: By definition, $AB = [A\vec{b}_1\ A\vec{b}_2\ A\vec{b}_3]$ (page 110).

(b) The second row of AB is the second row of A multiplied on the right by B.
Answer: True, but the row-column method of computing AB. See page 111.

(c) $(AB)C = (AC)B$.
Answer: False: it is NOT necessarily true that $BC = CB$, or even that the above products are both defined. See Theorem 2 and page 114.

(d) $(AB)^T = A^T B^T$.
Answer: False: $(AB)^T = B^T A^T$, by Theorem 3.

(e) The transpose of a sum of matrices equals the sum of their transposes.
Answer: True, by Theorem 3.

(2.2#7) Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\vec{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\vec{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

- Find A^{-1}, and use it to solve the four equations $A\vec{x} = \vec{b}_1$, $A\vec{x} = \vec{b}_2$, $A\vec{x} = \vec{b}_3$, $A\vec{x} = \vec{b}_4$.
Answer: The matrix A is 2×2, so its inverse is

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}.$$

For the first equation,

$$\vec{x} = A^{-1}\vec{b}_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}.$$

For the second equation,

$$\vec{x} = A^{-1}\vec{b}_2 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}.$$

For the third equation,

$$\vec{x} = A^{-1}\vec{b}_3 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

For the final equation,

$$\vec{x} = A^{-1}\vec{b}_4 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}.$$

- Solve the four equations in part (a) by row reducing the augmented matrix $[A \mid \vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4]$.
Answer: We have:

$$
\begin{aligned}
[A \mid \vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4] &= \left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{array}\right] \\
&\xrightarrow{R_2'=R_2-5R_1} \left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{array}\right] \\
&\xrightarrow{R_2'=R_2/2} \left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{array}\right] \\
&\xrightarrow{R_1'=R_1-2R_2} \left[\begin{array}{cc|cccc} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{array}\right].
\end{aligned}
$$

Reading off the solutions, we obtain the same solutions as in part (a).

(2.2#33) Use the algorithm from this section to find the inverses of $A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B, and then prove that $AB = I$ and $BA = I$.

Answer: We have:

$$
\begin{aligned}
[A_3|I_3] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2'=R_2-R_1, R_3'=R_3-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_3'=R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
[A_4|I_4] &= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2'=R_2-R_1, R_3'=R_3-R_1, R_4'=R_4-R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_3'=R_3-R_2, R_4'=R_4-R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_4'=R_4-R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right].
\end{aligned}
$$

In general, the inverse is

$$
B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 1 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}.
$$

(2.2#38) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Construct a 4×2 matrix D using only 1 and 0 as entries such that $AD = I_2$. Is it possible that $CA = I_4$ for some 4×2 matrix C? Why or why not?

Answer: A possible matrix D (in fact, the only possible matrix D) is $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Such a matrix C cannot exist, because $CA = [C\vec{a}_1\ C\vec{a}_2\ C\vec{a}_3\ C\vec{a}_4]$, and so since $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, and $\vec{a}_4$ are not linearly independent, neither are the columns of CA.

(2.3#13) When is a square upper triangular matrix invertible? Justify your answer.

Answer: A square upper triangular matrix is invertible if and only if it is row equivalent to the identity matrix, if and only if every position on the diagonal is a pivot position, which happens if and only if all of the diagonal entries of the matrix are nonzero.