Homework #7 Solution Set

March 30, 2006

(4.2#6) Find an explicit description of $NulA$ by listing vectors that span the null space for

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer: We have that

$$A \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so a general solution to $A\vec{x} = 0$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5.$$

Thus,

$$NulA = Span\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(4.2#16) Find A, where $ColA = \left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} | b, c, d \in \mathbb{R} \right\}$.

Answer: We have that:

$$\left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} b + \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} c + \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} d \right\} = Span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Thus, one possible A is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(4.3#14) Find bases for $NulA$ and $ColA$, where

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -5 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $A \sim B$.

Answer: A basis for $ColA$ is the pivot columns for A, so such a basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$. The general solution to $A\vec{x} = 0$ can be read off of an RREF form of A, so we row reduce B:

$$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution to $A\vec{x} = 0$ can now be read off, and is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ -7x_4/5 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 0 \\ -7/5 \\ 1 \\ 0 \end{bmatrix} x_4,$$

so a basis for $NulA$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(4.3#16) Find a basis for the space spanned by the following vectors $\vec{v}_1, \dots, \vec{v}_5$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

Answer: We know that there must be some linear dependence relation among these vectors, as there are 5 vectors but only 4 entries per vector Using the terminology of Section 4.5, the dimension of the span of these vectors can be at most 4, so this set cannot be a basis. That means there is a nontrivial solution to $c_1\vec{v}_1 + \cdots + c_5\vec{v}_5 = \vec{0}$, which means that the matrix $[\vec{v}_1 \dots \vec{v}_5]$ corresponds to a system of linear equations which has free variables. In other words, there are nonpivot columns in $[\vec{v}_1 \dots \vec{v}_5]$, so not all of $\vec{v}_1, \dots, \vec{v}_5$ are needed to span $Col[\vec{v}_1 \dots \vec{v}_5] = Span\{\vec{v}_1, \dots, \vec{v}_5\}$. We row reduce $[\vec{v}_1 \dots \vec{v}_5]$ to find its pivot columns and thus a basis for its column space:

$$[\vec{v}_1 \dots \vec{v}_5] = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for this vector space is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(4.3#21) (a) A single vector by itself is linearly dependent.

Answer: False: only if the vector is $\vec{0}$.

(b) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

Answer: True: this is the invertible matrix theorem.

(c) A basis is a spanning set that is as large as possible.

Answer: False: A basis must also be linearly independent.

(d) In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

Answer: False: linear dependence relations among columns are unaffected by row operations.

(4.3#36) Let $H = Span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $K = Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 2 \\ 2 \\ 7 \\ -3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 8 \\ -4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 9 \\ -5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 6 \\ -2 \end{bmatrix}.$$

Answer:

A basis for $H = Span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = Span[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$ is $\{\vec{u}_1, \vec{u}_2\}$, as these are the pivot columns of $[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$:

$$[\vec{u}_1\ \vec{u}_2\ \vec{u}_3] = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & -1 & 7 \\ -1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for $K = Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = Span[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ is $\{\vec{v}_1, \vec{v}_2\}$, as these are the pivot columns of $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$:

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 8 & 9 & 6 \\ -4 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $H = Span\{\vec{u}_1, \vec{u}_2\}$ and $K = Span\{\vec{v}_1, \vec{v}_2\}$,

$$H + K = Span\{\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2\} = Col[\vec{u}_1\ \vec{u}_2\ \vec{v}_1\ \vec{v}_2].$$

A basis for $H + K$ is $\{\vec{u}_1, \vec{u}_2, \vec{v}_1\}$, as these are the pivot columns of $[\vec{u}_1\ \vec{u}_2\ \vec{v}_1\ \vec{v}_2]$:

$$[\vec{u}_1\ \vec{u}_2\ \vec{v}_1\ \vec{v}_2] = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & -2 \\ 3 & -1 & 8 & 9 \\ -1 & 1 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$