Homework #8 Solution Set

April 6, 2006

(4.5#6) Let

$$H = \left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} | a,b,c \in \mathbb{R} \right\}.$$

Find a basis for H and state its dimension.

Answer: We have:

$$H = \left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} | a,b,c \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} a + \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} b + \begin{bmatrix} - \\ -2 \\ 3 \\ 1 \end{bmatrix} c | a,b,c \in \mathbb{R} \right\}$$

$$= Span \left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Let A be the matrix whose columns are the three vectors in the given span. Then

$$A = \begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $H = Col\ A$ has as a basis the pivot columns of A, a basis for H is $\left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} \right\}$. The dimension of this space is 2.

(4.5#21) The first four Hermite polynomials are 1, $2t$, $-2+4t^2$, and $-12t+8t^3$. Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

Answer: Since $\dim \mathbb{P}_3 = 4$ and we have four polynomials, it suffices to prove that either the four Hermite polynomials are linearly independent or that they span $\mathbb{P}_3$, by the Basis Theorem. But the first four Hermite polynomials are clearly linearly independent: if they were linearly dependent, then one of them would be able to be written as a linear combination of the previous polynomials (by Theorem 4), which cannot happen as each successive polynomial involves a higher and higher power of t.

(4.5#23) Let $\mathcal{B}$ be the basis of $\mathbb{P}_3$ consisting of the first four Hermite polynomials, and let $\mathbf{p}(t) = 7-12t-8t^2+12t^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

Answer: Since $\mathcal{B}$ is a basis for $\mathbb{P}_3$, we know that $mathbfp$ may be written uniquely as a linear combination of the Hermite polynomials:

$$c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) = (7-12t-8t^2+12t^3).$$

We need to find c_1, c_2, c_3, and c_4. One way is to use matrices:

$$c_1 \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}2\\2\\0\\0\end{bmatrix} + c_3 \begin{bmatrix}-2\\0\\4\\0\end{bmatrix} + c_4 \begin{bmatrix}0\\-12\\0\\8\end{bmatrix} = \begin{bmatrix}7\\-12\\-8\\12\end{bmatrix}.$$

Row reducing the augmented matrix corresponding to this vector equation, we get that $c_1 = 3$, $c_2 = 3$, $c_3 = -2$, and $c_4 = 3/2$.

Another way is to look at the coefficients of each power of t in the original equation:

$$(c_1 - 2c_3) + (2c_2 - 12c_4)t + (4c_3)t^2 + (8c_4)t^3 = (7) + (-12)t + (-8)t^2 + (12)t^3.$$

Making the appropriate coefficients match, we obtain the system of linear equations:

$$\begin{array}{ccccc} c_1 & & -2c_3 & & = 7 \\ & 2c_2 & & -12c_4 & = -12 \\ & & 4c_3 & & = -8 \\ & & & 8c_4 & = 12 \end{array}.$$

Solving this system of linear equations, we get the same answer as above.

(4.6#3) Define

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $A \sim B$. List $rank\ A$ and $\dim Nul\ A$. Find bases for $Col\ A$, $Row\ A$, and $Nul\ A$.

Answer: By the Rank Theorem, $rank\ A$ is the number of pivot columns of A, which in this case is 3. Also by the Rank Theorem, $\dim Nul\ A = n - rank\ A = 5 - 3 = 2$.

A basis for $Col\ A$ is the pivot columns of A, which is in this case

$$\left\{ \begin{bmatrix}2\\-2\\4\\-2\end{bmatrix}, \begin{bmatrix}6\\-3\\9\\3\end{bmatrix}, \begin{bmatrix}2\\-3\\5\\-4\end{bmatrix} \right\}.$$

A basis for $Row\ A$ is the nonzero rows of a REF for A - for instance, the nonzero rows of B:

$$\left\{ \begin{bmatrix}2 & -3 & 6 & 2 & 5\end{bmatrix}, \begin{bmatrix}0 & 0 & 3 & -1 & 1\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & 1 & 3\end{bmatrix} \right\}.$$

To find a basis for $Nul\ A$, we must solve the homogenous system corresponding to $A\vec{x} = \vec{0}$, so we must continue row reducing B to get to a RREF form:

$$B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & 0 & -1 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Reading off the general solution, we have:

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}3x_2/2 + 9x_5/2\\x_2\\-4x_5/3\\-3x_5\\x_5\end{bmatrix} = \begin{bmatrix}3/2\\1\\0\\0\\0\end{bmatrix} x_2 + \begin{bmatrix}9/2\\0\\-4/3\\-3\\1\end{bmatrix} x_5.$$

Thus, a basis for $Nul\ A$ is

$$\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

(4.6#8) Suppose a 5×6 matrix A has four pivot columns. What is $\dim Nul\ A$? Is $Col\ A = \mathbb{R}^4$?

Answer: As A has four pivot columns, $rank\ A = 4$. By the Rank Theorem, $\dim Nul\ A = n - rank\ A = 6 - 4 = 2$. The space $Col\ A$ is a subspace of $\mathbb{R}^5$, and so cannot be $\mathbb{R}^4$, even though it is 4 dimensional.

(5.1#7) Is $\lambda = 4$ an eigenvalue of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

Answer: If $\lambda = 4$ is an eigenvalue of A, then there is a nontrivial solution to the equation $A\vec{x} = \lambda\vec{x}$ - i.e. $(A - \lambda I)\vec{x} = \vec{0}$. Row reducing $A - \lambda I$, we see that

$$[A - \lambda I] = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there is a nonpivot column in $A - \lambda I$, there will be a free variable in the solution to $(A - \lambda I)\vec{x} = \vec{0}$. The free variable will give us nontrivial solutions, so $\lambda = 4$ is an eigenvalue of A. to find an eigenvector, we read off the general solution to $(A - \lambda I)\vec{x} = \vec{0}$: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$. An eigenvector is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(5.1#33) Let $\vec{u}$ and $\vec{v}$ be eigenvectors of a matrix A, with corresponding eigenvalues λ and μ, and let c_1 and c_2 be scalars. Define

$$\mathbf{x}_k = c_1\lambda^k\vec{u} + c_2\mu^k\vec{v} \quad (k = 0, 1, 2, \dots)$$

(a) What is $\mathbf{x}_{k+1}$, by definition?
Answer: We have:

$$\mathbf{x}_{k+1} = c_1\lambda^{k+1}\vec{u} + c_2\mu^{k+1}\vec{v}$$

(b) Compute $A\mathbf{x}_k$ from the formula for $\mathbf{x}_k$, and show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$.
Answer: Since $\vec{u}$ and $\vec{v}$ are eigenvectors of A with corresponding eigenvalues λ and μ, we have:

$$A\mathbf{x}_k = c_1\lambda^k A\vec{u} + c_2\mu^k A\vec{v} = c_1\lambda^k\lambda\vec{u} + c_2\mu^k\mu\vec{v} = c_1\lambda^{k+1}\vec{u} + c_2\mu^{k+1}\vec{v} = \mathbf{x}_{k+1}.$$