

Math 601B Homework # 4
Due Wednesday, April 14

- [Bridson-Haefliger] Let X be the closed subset of \mathbb{E}^3 that is the complement of the octant $\{(x, y, z) : x > 0, y > 0, z > 0\}$. Show that X , endowed with the induced length metric, is *not* a CAT(0) space (or indeed a CAT(κ) space for any κ).
- [Bridson-Haefliger] Let X be a CAT(κ) space and fix $x_0 \in X$. Prove that the restriction of $x \mapsto d(x, x_0)$ to the open ball of radius $D_\kappa/2$ about x_0 is convex.
- An \mathbb{R} -tree is a metric space T such that:
 - there is a *unique* geodesic segment (denoted $[x, y]$) joining each pair of points $x, y \in T$, and
 - if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Let X be a geodesic metric space. Prove that the following are equivalent:

- X is an \mathbb{R} -tree
- X is 0-hyperbolic
- X is CAT(κ) for every κ .

- First, we give some definitions.

A cell complex X is a *cubical complex* if the cells of X are isometric copies of the cubes $[0, 1]^n$, and any two cells are disjoint or meet in a single face.

If v is a vertex of a cubical complex X , then $lk(v, X)$ is the simplicial complex that has one n -simplex corresponding to every $(n+1)$ -cell of X that contains v . A simplex σ in $lk(v, X)$ is a face of a simplex σ' if and only if the cell of X corresponding to σ is a face of the cell corresponding to σ' .

A simplicial complex X satisfies (*) if, for every vertex v in X , whenever every pair of vertices in a subset of $(n+1)$ vertices of $lk(v, X)$ is connected by an edge, then the set spans an n -simplex. (I.e. wherever there is the 1-skeleton of an n -simplex, there is also the n -simplex.)

- Give an example of a simply connected cubical complex that satisfies (*).
- Give an example of a simply connected cubical complex that does not satisfy (*).
- Give an example of a cubical complex that satisfies (*) and is not simply connected.

Remind me on Wednesday to tell you about the connection, due to Gromov, between the condition (*) and CAT(0) spaces.

