Group actions on trees: supplement[*]

Note to the reader: be sure to draw yourself a picture with these arguments! If I were better at tex I would put some in myself.

Let G be a group, T a simplicial metric tree, where we take all edges to be isometric copies of $[0,1]$, and assume that we are given an action of G on T without inversions. For any $p, q \in T$, let $[p,q]$ denote the (unique) geodesic segment in T between p and q.

Definition 1. *Let $g \in G$, $l \subset T$. We say that l is an* axis *for g if l is a bi-infinite geodesic (a "line"), and g acts on l by (nontrivial) translation.*

We will make use of the following observation throughout these notes.

Remark 2. *Let $x \in T$, let $g \in G$ such that g does not fix a point of T, and let $l = \cup_{n\in\mathbb{Z}} g^n[x, gx] = \cup_{n\in\mathbb{Z}}[g^n x, g^{n+1}x]$. Clearly l is g-invariant and connected, hence a subtree of T. Thus l is an axis for g if and only if the following equivalent conditions hold:*

1. *l is a line*
2. *the adjacent segments $[g^{n-1}x, g^n x], [g^n x, g^{n+1}x]$ meet only in the point $g^n x$ for some, or equivalently every, n*
3. *$[g^{n-1}x, g^{n+1}x] = [g^{n-1}x, g^n x] \cup [g^n x, g^{n+1}x]$ for some, equivalently every, n*
4. *$d(g^{n-1}x, g^{n+1}x) = d(g^{n-1}x, g^n x) + d(g^n x, g^{n+1}x)$ for some, equivalently every, n*
5. *$d(x, g^2x) = 2d(x, gx)$*

For the proof of the following lemma, we follow Henry Wilton's notes.

Lemma 3. *Fix $g \in G$. Either g fixes a vertex of T or g has an axis.*

Proof. Suppose that g fixes no vertex of T. Since G acts on T without inversions, it follows that g fixes no point of T.

Fix any point $v \in T$. If $[v, gv] \cap [gv, g^2v] = gv$, then $\cup g^n[v, gv]$ is an axis for g (see Remark 2), so we are done. Suppose instead that $[v, gv]$ and $[gv, g^2v]$ do

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not meet only at gv. Then they form a nondegenerate tripod; let p be the point that separates this tripod into three connected components, and let m be the midpoint of $[v, gv]$.

If we had $d(v,p) \leq d(v,m)$, then it must be the case that $m \in [gv, g^2v]$. Since $d(m, gv) = \frac{1}{2}\text{length}[v, gv] = \frac{1}{2}\text{length}[gv, g^2v]$, m is also the midpoint of $[gv, g^2v]$. As g takes $[v, gv]$ to $[gv, g^2v]$ isometrically, it would follow that g fixes m, a contradiction. Thus $d(v,p) > d(v,m)$.

We will show that $[m, gm]$ and $[gm, g^2m]$ meet only at gm. Since $p \in [m, gm]$, it suffices to show that $[p, gm]$ and $[gm, gp]$ meet only at gm, or, equivalently, that $d(p, gp) = 2d(p, gm)$. Note that $[gv, g^2v]$ can be decomposed into the three segments $[gv, p]$, $[p, gp]$ and $[gp, g^2v]$, which have disjoint interiors. Note also that $d(m,p) = d(p, gm)$, for $[m, gv]$ and $[gv, gm]$ overlap precisely on $[p, gv]$. Thus we have

$$d(p, gp) = d(gv, g^2v) - (d(p, gv) + d(gp, g^2v)) = d(v, gv) - 2d(p, gv)$$

$$= d(v, gv) - 2(d(m, gv) - d(m,p)) = d(v, gv) - 2(\frac{1}{2}d(v, gv) - d(p, gm))$$

$$= 2d(p, gm),$$

as desired. ■

In our proof, we also saw the following.

Corollary 4. *If $g \in G$ does not fix any vertex of T and $v \in T$, then $[v, gv]$ meets an axis for g at its midpoint, and moreover $\cup_{n \in \mathbb{Z}} g^n[v, gv]$ contains that axis.*

Definition 5. *For $g \in G$, define*

$$l(g) = \inf_{p \in T} d(p, gp).$$

Recall that we have metrized T so that all edges are of length 1.

Lemma 6. *For any $g \in G$, $l(g) \in \mathbb{Z}$ and there exists $p \in T$ such that $d(p, gp) = l(g)$.*

Let us fix some more notation, for use in the proof. If e is an oriented edge of T, then let $i(e)$ and $t(e)$ denote the initial and terminal vertices of e respectively. For any connected $S \subset T$ that is disjoint from e, we shall say that e is oriented *towards* S if $t(e)$ is contained in the unique geodesic segment from e to S. Otherwise (that is, if $i(e)$ is contained in that segment), we shall say that e is oriented *away* from S.

Proof of Lemma 6. The lemma is clear if g fixes a point of T. Suppose instead that g does not fix a point, and we shall show that in this case, for any $p \in T$, either $d(p, gp) \in \mathbb{Z}$ or there is some $v \in T$ such that $d(v, gv) \in \mathbb{Z}$ and $d(v, gv) < d(p, gp)$. From this, the lemma follows.

Clearly if p is a vertex of T, then $d(p, gp) \in \mathbb{Z}$. Assume then that p is not a vertex of T.

Let e be the edge containing p. As G acts on T without inversions, note that e and ge are distinct, as unoriented edges. Consider e now with orientation towards gp. If ge is oriented towards e, then certainly $gt(e) = t(ge)$ and hence $d(t(e), gt(e)) < d(p, gp)$. As $t(e)$ is a vertex of T, we have $d(t(e), gt(e)) \in \mathbb{Z}$.

If ge is oriented away from e, then we have $t(e), i(ge) \in [p, gp]$. Let α be the distance from p to $t(e)$, and it follows that $d(p, gp) = d(p, t(e)) + d(t(e), i(ge)) + d(i(ge), gp) = \alpha + d(t(e), i(ge)) + (1 - \alpha)$. As $t(e)$ and $i(ge)$ are vertices, $d(t(e), i(ge)) \in \mathbb{Z}$ and hence $d(p, gp) \in \mathbb{Z}$. ■

Lemma 7. *If $g \in G$ does not fix a vertex of T, then it has a unique axis given by*

$$l = \{p \in T : d(p, gp) = l(g)\}.$$

Proof. We shall first see that l contains an axis of g. The segments $[v, gv]$ and $[gv, g^2v]$ meet only at gv if and only if $d(m, gm) = d(v, gv)$, if m denotes the midpoint of $[v, gv]$. Otherwise $d(m, gm) < d(v, gv)$. Hence if $v \in l$ then $d(v, gv)$ is minimal and hence $[v, gv]$ and $[gv, g^2v]$ meet only at gv, so, as we saw in Remark 2, $l_v = \cup g^n[v, gv] \subset l$ is an axis for g.

Note that, if l' is any axis of g, and $v \in l'$, then $l' = \cup g^n[v, gv]$. Now suppose that $l_w = \cup g^n[w, gw]$ is another axis for g. If $l_v \cap l_w \neq \emptyset$, then fix $p \in l_v \cap l_w$ and we have that both l_v and l_w are equal to $\cup g^n[p, gp]$, so $l_v = l_w$.

Suppose then that $l_v \cap l_w = \emptyset$. Recall that, in a tree, there is a unique geodesic segment connecting any two connected, disjoint regions. Let γ be the geodesic between l_v and l_w, and consider $g\gamma$, which must also be a geodesic segment connecting l_v and l_w. It follows that $\gamma = g\gamma$ and in particular that g fixes the endpoints of γ, a contradiction.

Thus $l = l_v$ is the unique axis for g. ■

The next corollary is immediate from Lemma 7.

Corollary 8. *If $g \in G$ has an axis, then g acts on that axis by translation of length $l(g)$.*

Let $fix(g) = \{p \in T : p = gp\}$. If $fix(g)$ is nonempty, then note that it is connected, hence a subtree of T.

Lemma 9. *If $g, h \in G$ fix vertices of T and $fix(g) \cap fix(h) = \emptyset$, then gh has an axis on which gh acts by a translation of length $2d$, where d is the length of the geodesic segment connecting $fix(g)$ to $fix(h)$.*

Proof. Let γ denote the geodesic segment from $fix(g)$ to $fix(h)$, and let p_g, p_h denote the endpoints of γ, with $p_g \in fix(g)$ and $p_h \in fix(h)$.

As γ meets $fix(g)$ only at p_g, $g\gamma$ must meet γ only at p_g. It follows that $\gamma \cup g\gamma = [p_h, gp_h = ghp_h]$, and hence that $d(p_h, ghp_h) = 2d$. We also have that $\gamma \cap h\gamma = p_h$, hence $g\gamma \cap gh\gamma = gp_h$.

Similarly $\gamma \cap h^{-1}\gamma = p_h$, so $g\gamma \cap h^{-1}\gamma = \emptyset$ since $p_g \in g\gamma, p_h \in h^{-1}\gamma$ and any path from p_g to p_h must contain γ. Translation by gh gives that $ghg\gamma \cap g\gamma = \emptyset$.

Making the same argument, with $h\gamma$ replacing γ, gives also that $gh\gamma \cap \gamma = \emptyset$. Also a similar argument gives $\gamma \cap ghg\gamma = \emptyset$.

It follows that $(\gamma \cup g\gamma) = [p_h, (gh)p_h]$ meets $(gh\gamma \cup ghg\gamma) = (gh)(\gamma \cup g\gamma) = [(gh)p_h, (gh)^2 p_h]$ only at $(gh)p_h$. Hence, by Remark 2, $\cup (gh)^n(\gamma \cup g\gamma)$ is the axis for gh. The element gh has translation length $2d$ on the axis, thus Lemma 7 implies $2d = l(g)$. ■