Solutions for HW1.

Ex. 5.1.1.

Proof: Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy seq. $\Rightarrow$ for $\varepsilon=1$, $\exists\ N\geq 1$,

s.t. $|a_n - a_N| \leq 1$, $\forall\ n\geq N$.

$\Rightarrow$ $|a_n| \leq |a_N|+1$, $\forall\ n \geq N$.

and $\{a_n\}_{n=1}^{N}$ is a finit seq. $\Rightarrow$ $\{a_n\}_{n=1}^{N}$ are bounded.

i.e. $|a_n| \leq M_1$, for $1\leq n \leq N$. for some $M_1 > 0$.

Hence: $|a_n| \leq M_1 + |a_N| + 1$, $\forall\ n\geq 1$. #.

Ex. 5.2.1.

Pf: Since $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are equivalent seqs.

① $\{a_n\}_{n=1}^{\infty}$ Cauchy $\Rightarrow$ $\{b_n\}_{n=1}^{\infty}$ Cauchy.

Need only to show: $\forall\ \varepsilon>0$, $\exists\ N\geq 1$,

$|b_i - b_j| \leq \varepsilon$, $\forall\ i,j\geq N$.

$\because$ $\{a_n\}_{n=1}^{\infty}$ Cauchy $\Rightarrow$ For $\frac{\varepsilon}{2}>0$, $\exists\ N_1\geq 1$.

$|a_i - a_j| \leq \frac{\varepsilon}{2}$, $\forall\ i,j\geq N_1$

and $\{a_n\}$ & $\{b_n\}$ equivalent $\Rightarrow$ For $\frac{\varepsilon}{4}>0$ $\exists\ N_2\geq 1$

$|a_n - b_n| \leq \frac{\varepsilon}{4}$, $\forall\ n\geq N_2$.

Take $N = N_1 + N_2$, for $i,j\geq N$

$$|b_i - b_j| = |b_i - a_i + a_i - a_j + a_j - b_j| \leq |a_i - b_i| + |a_i - a_j| + |a_j - b_j|$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

② $\{b_n\}_{n=1}^{\infty}$ Cauchy $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ Cauchy.

Same as ①. #

Solutions for HW 1

Ex. 5.2.2.

<u>Proof</u>: $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are eventually ε-close.

① $\{a_n\}_{n=1}^{\infty}$ bounded $\Rightarrow$ $\{b_n\}_{n=1}^{\infty}$ bounded.

Need only to show: $\exists M_b > 0$. s.t. $|b_n| \le M_b$. $\forall n \ge 1$.

$\because$ $\{a_n\}$ & $\{b_n\}$ eventually ε-close $\Rightarrow$

$\exists N \ge 1$, s.t. $|a_n - b_n| \le \varepsilon$, $\forall n \ge N$.

and $\{a_n\}$ bounded $\Rightarrow$ $\exists M_a > 0$. s.t. $|a_n| \le M_a$, $\forall n \ge 1$.

$\Rightarrow$ For $n \ge N$, $|b_n| \le |a_n - b_n| + |a_n| \le M_a + \varepsilon$,

For $1 \le n \le N$, $\{b_n\}_{n=1}^{N}$ is bounded, since finit seq.

$\Rightarrow$ $\exists M_1 \ge 0$. s.t. $|b_n| \le M_1$ for $1 \le n \le N$.

Take $M_b = M_1 + M_a + \varepsilon$, we have

$|b_n| \le M_b$, for all $n \ge 1$.

② $\{b_n\}_{n=1}^{\infty}$ bounded $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ bounded.

same as ①. #

Ex: 5.3.2.

<u>Proof</u>. ① Need to show: For Cauchy seqs $\{a_n\}$, $\{b_n\}$, $\{a_n b_n\}$ Cauchy too.

First $\{a_n\}$ & $\{b_n\}$ both bounded, i.e. $\exists M > 0$.

s.t. $|a_n| \le M$, $|b_n| \le M$ $\forall n \ge 1$

For given $\varepsilon > 0$, $\{a_n\}$, $\{b_n\}$ are eventually $\frac{\varepsilon}{2M}$-steady.

$\Rightarrow$ $\exists N \ge 1$, s.t. for any $i, j \ge N$,

$|a_i - a_j| \le \frac{\varepsilon}{2M}$, $|b_i - b_j| \le \frac{\varepsilon}{2M}$.

Now $|a_i b_i - a_j b_j| = |a_i b_i - a_i b_j + a_i b_j - a_j b_j|$

$\le |a_i|\,|b_i - b_j| + |b_j|\,|a_i - a_j| \le M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon$.

$\Rightarrow$ $\{a_n b_n\}$ is eventually ε-steady $\forall \varepsilon > 0$

$\Rightarrow$ $\{a_n b_n\}$ Cauchy.

Solutions of HW 1

(Continue) ② If $\{a_n\}$ and $\{a_n'\}$ equivalent Cauchy seq, $\{b_n\}$ Cauchy seq.
$\Rightarrow$ $\{a_n b_n\}$ and $\{a_n' b_n\}$ equivalent Cauchy seq.

Need to show $\{a_n b_n\}$ & $\{a_n' b_n\}$ are eventually ε-close, $\forall \varepsilon > 0$.

$\{b_n\}$ Cauchy $\Rightarrow$ $\{b_n\}$ bounded, i.e. $\exists M > 0$, $|b_n| \le M$, $\forall n \ge 1$.

Given $\varepsilon > 0$, $\{a_n\}$ & $\{a_n'\}$ equivalent $\Rightarrow$ eventually $\frac{\varepsilon}{M}$-close

i.e. $\exists N \ge 1$, s.t. $|a_n - a_n'| \le \frac{\varepsilon}{M}$, $\forall n \ge N$.

$\Rightarrow |a_n b_n - a_n' b_n| = |a_n - a_n'| \cdot |b_n| \le \frac{\varepsilon}{M} \cdot M = \varepsilon$. $\forall n \ge N$.

$\Rightarrow$ $\{a_n b_n\}$ & $\{a_n' b_n\}$ are eventually ε-close. $\forall \varepsilon > 0$

$\Rightarrow$ $\{a_n b_n\}$ & $\{a_n' b_n\}$ are equivalent. #

Ex. 5.3.3. a, b rationals.

Proof. ① $a = b$ $\Rightarrow$ $\{a\}_{n=1}^{\infty}$ and $\{b\}_{n=1}^{\infty}$ equivalent.

By definition, it is true since $|a-b| = 0$.

② $\{a\}_{n=1}^{\infty}$ and $\{b\}_{n=1}^{\infty}$ equivalent $\Rightarrow$ $a = b$.

For any $\varepsilon > 0$, $\{a\}_{n=1}^{\infty}$ & $\{b\}_{n=1}^{\infty}$ are eventually ε-close

$\Rightarrow |a-b| \le \varepsilon$, $\Rightarrow |a-b| = 0 \Rightarrow a = b$. #.

Ex. 5.3.4. $\{a_n\}$ bounded, $\{a_n\}$ and $\{b_n\}$ are equivalent

Proof: $\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ are eventually ε-close $\forall \varepsilon > 0$.

By ex 5.2.2. $\Rightarrow$ $\{b_n\}$ bounded. #

Ex. 5.3.5

Proof. Need to show $\{\frac{1}{n}\}_{n=1}^{\infty}$ and $\{0\}_{n=1}^{\infty}$ are equivalent.

Given $\varepsilon > 0$, take $N > 0$, s.t. $\frac{1}{N} \le \varepsilon$,

For $n \ge N$, we have

$|\frac{1}{n} - 0| = \frac{1}{n} \le \frac{1}{N} \le \varepsilon$.

$\Rightarrow$ $\{\frac{1}{n}\}_{n=1}^{\infty}$ and $\{0\}_{n=1}^{\infty}$ equivalent $\Rightarrow$ $\mathrm{LIM}_{n\to\infty} \frac{1}{n} = 0$. #