Solutions of HW2

Ex. 5.4.1.

Proof. ①. Exactly one of statements true: (a) $x=0$. (b) x is positive (c) x is negative.

Assume $x \neq 0$, $x = \operatorname{LIM}_{n\to\infty} a_n$, $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ and $\{0\}$ are not equivalent.

$\Rightarrow \exists\, \varepsilon > 0$. $\{a_n\}_{n=1}^{\infty}$ and $\{0\}_{n=1}^{\infty}$ are not eventually ε-close.

This means, for any $N \geq 1$, $\exists\, n_N \geq N$, s.t.

$|a_{n_N}| > \varepsilon > 0$.

Since $\{a_n\}$ Cauchy. $\Rightarrow$ $\{a_n\}$ is eventually $\frac{\varepsilon}{2}$-steady.

$\Rightarrow \exists\, N_1 \geq 1$. s.t. $\forall\, i, j \geq N_1$, $|a_i - a_j| \leq \frac{\varepsilon}{2}$.

Pick $n_{N_1} \geq N_1$, s.t. $|a_{n_{N_1}}| > \varepsilon > 0$.

(a). $a_{n_{N_1}} > \varepsilon$, then $\forall\, n \geq N_1$

$$a_n = a_n - a_{n_{N_1}} + a_{n_{N_1}} \geq a_{n_{N_1}} - |a_n - a_{n_{N_1}}| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0.$$

i.e. $\{a_n\}_{n=N_1}^{\infty}$ positively bounded away from zero.

$\Rightarrow$ $x = \operatorname{LIM}_{n\to\infty} a_n$ is positive.

(b) $a_{n_{N_1}} < -\varepsilon$. same arguement $\Rightarrow$

$\{a_n\}_{n=N_1}^{\infty}$ negatively bounded away from zero

$\Rightarrow$ $x = \operatorname{LIM}_{n\to\infty} a_n$ is negatively.

(a) and (b) Can't happen both

$\Rightarrow$ x positive or negative can only be one. #

②. x negative iff $-x$ positive

x negative $\Leftrightarrow$ $x = \operatorname{LIM}_{n\to\infty} a_n$, $\{a_n\}$ negatively bounded away from zero

$\Leftrightarrow \exists\, c > 0$. $a_n \leq -c$, $\forall\, n \geq 1$ $\Leftrightarrow$ $-a_n \geq c > 0$, $\forall\, n \geq 1$.

$\Leftrightarrow$ $\{-a_n\}$ positively bounded away from zero.

$\Leftrightarrow$ $-x = \operatorname{LIM}_{n\to\infty}(-a_n)$ positive.

③. x, y positive $\Rightarrow$ $x+y$ and xy positive.

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(Continue). x, y positive $\Rightarrow \exists\, C>0$, $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, $a_n \geq C$, $b_n \geq C$, $\forall n \geq 1$.

$\Rightarrow x+y = \mathrm{LIM}_{n\to\infty}(a_n+b_n)$, $a_n+b_n \geq 2C$, $\{a_n+b_n\}$ positively bounded away from zero

$xy = \mathrm{LIM}_{n\to\infty} a_n b_n$, $a_n b_n \geq C^2 > 0$, $\{a_n b_n\}$ positively bounded away from zero

$\Rightarrow x+y,\ xy$ positive. #

Ex. 5.4.4.

Proof: Since x positive, $\Rightarrow \exists\, C>0$, $x = \mathrm{LIM}_{n\to\infty} a_n$ and $a_n \geq C$, $\forall n \geq 1$.

Choose $N \geq 1$ large enough, s.t. $\frac{1}{N} < C$.

$\Rightarrow a_n \geq C > \frac{1}{N} \quad \forall n \geq 1$.

$\Rightarrow x = \mathrm{LIM}_{n\to\infty} a_n \geq \mathrm{LIM}_{n\to\infty} C > \mathrm{LIM}_{n\to\infty} \frac{1}{N}$

$\Rightarrow x \geq C > \frac{1}{N} \geq 0$. #.

Ex. 5.4.5. reals $x<y$, $\Rightarrow \exists$ rational q, s.t. $x < q < y$.

Proof: Since $y - x > 0$. By Ex 5.4.4, $\exists\, K \geq 1$, s.t.

$y - x > \frac{1}{K}$, Fix this K.

let $x = \mathrm{LIM}_{n\to\infty} a_n$, $\{a_n\}$ Cauchy seq of rationals.

For $\varepsilon = \frac{1}{4K} > 0$, $\{a_n\}$ is eventually ε-steady. i.e. $\exists\, N \geq 1$, s.t.

$|a_i - a_j| \leq \varepsilon. \quad \forall i, j \geq N.$

$\Leftrightarrow -\frac{1}{4K} = -\varepsilon \leq a_i - a_j \leq \varepsilon = \frac{1}{4K} \qquad (*)$

Claim: $x < a_N + \frac{1}{2K} \leq x + \frac{1}{K} < y$

From $(*) \Rightarrow \quad a_i \leq a_N + \frac{1}{4K} \quad \forall i \geq N. \Rightarrow x = \mathrm{LIM}_{i\to\infty} a_i \leq a_N + \frac{1}{4K} < a_N + \frac{1}{2K}$

$\Rightarrow a_i \geq a_N - \frac{1}{4K} \ \forall i \geq N \Rightarrow x = \mathrm{LIM}_{i\to\infty} a_i \geq a_N - \frac{1}{4K}$

$\Rightarrow x + \frac{1}{K} \geq a_N + \frac{3}{4K} > a_N + \frac{1}{2K}$

Hence choose $q = a_N + \frac{1}{2K}$ rational.

$x < q < y$ are true. #

Solutions of HW 2

Ex. 5.4.7 x, y reals, $\varepsilon > 0$ real.

Proof: ①. $x \le y + \varepsilon$, $\forall \varepsilon > 0$ iff $x \le y$.

ⓐ $x \le y + \varepsilon$, $\forall \varepsilon > 0 \Rightarrow x \le y$.

using contradiction argument.

Assume $x > y$, from Ex 5.4.4 $\exists N > 0$ s.t. $x - y > \frac{1}{N} > 0$.

Then for $\varepsilon = \frac{1}{N} > 0$, $x > y + \frac{1}{N}$, contradiction.

ⓑ. $x \le y \Rightarrow x \le y + \varepsilon$, $\forall \varepsilon > 0$.

two Cases: $x = y$ or $x < y$; Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$.

For $x = y \Rightarrow \{a_n\}$ & $\{b_n\}$ equivalent

Given $\varepsilon > 0$; $\{a_n\}$ & $\{b_n\}$ are eventually $\frac{\varepsilon}{2}$-close.

$\Rightarrow \exists N \ge 1$, s.t. $|a_n - b_n| \le \frac{\varepsilon}{2}$, $\forall n \ge N$.

$\Rightarrow b_n + \varepsilon - a_n \ge \varepsilon - |a_n - b_n| \ge \frac{\varepsilon}{2} > 0$.

$\Rightarrow \{b_n + \varepsilon - a_n\}_{n \ge N}^{\infty}$ are positively bounded away from zero.

$\Rightarrow \mathrm{LIM}_{n\to\infty}(b_n + \varepsilon - a_n) > 0 \Rightarrow y + \varepsilon - x > 0 \Rightarrow x < y + \varepsilon$.

For $x < y$, $\Rightarrow y - x > 0 \Rightarrow \exists \{c_n\}$ positively bounded away from zero

s.t. $y - x = \mathrm{LIM}_{n\to\infty} c_n = \mathrm{LIM}_{n\to\infty}(b_n - a_n)$

i.e. $\exists c > 0$, s.t. $c_n \ge c > 0$ $\forall n \ge 1$.

$\Rightarrow \mathrm{LIM}_{n\to\infty} c_n \ge c > 0$.

For $\forall \varepsilon > 0$,

$y + \varepsilon - x = \mathrm{LIM}_{n\to\infty}(b_n + \varepsilon - a_n) = \mathrm{LIM}_{n\to\infty}(c_n + \varepsilon) \ge c + \varepsilon > 0$

$\Rightarrow x < y + \varepsilon$. #

②. $|x - y| \le \varepsilon$, $\forall \varepsilon > 0$ iff $x = y$.

$|x - y| \le \varepsilon$, $\forall \varepsilon > 0 \Leftrightarrow y - \varepsilon \le x \le \varepsilon + y$, $\forall \varepsilon > 0$. from Ex. 5.4.6

$\Leftrightarrow \begin{matrix} x \le y + \varepsilon, \forall \varepsilon > 0 \\ \text{and } y \le x + \varepsilon, \forall \varepsilon > 0 \end{matrix} \Leftrightarrow \begin{cases} x \le y \\ \text{and } y \le x \end{cases} \Leftrightarrow x = y$. #

Solutions of HW 2.

Ex. 5.5.2.

Pf: Induction on $k = K - L$, for ~~any~~ each $n \ge 1$.

step 1. $k = 1$, i.e. $L = K-1$, we choose $m = K$.

$\frac{K}{n} = \frac{m}{n}$ is an upper bound and $\frac{K-1}{n} = \frac{L}{n}$ is not an upper bound.

step 2: Assume for $k = K-L$, we can find $m \le K$, s.t.

$\frac{m}{n}$ is an upper bound and $\frac{m-1}{n}$ is not an upper bound

Then for $k+1 = K-L$, there are two cases:

①. $m = K$, i.e. $\frac{K}{n}$ is an upper bound and $\frac{K-1}{n}$ is not an upper bound then we finish the proof.

② otherwise $m \ne K$. i.e. $\frac{K-1}{n}$ is also an upper bound.

Now we consider $\frac{K-1}{n}$ and $\frac{L}{n}$.

$\frac{K-1}{n}$ is an upper bound and $\frac{L}{n}$ is not an upper bound

$K-1-L = k$. by induction assume

we can find m, $L < m \le K-1 < K$, s.t.

$\frac{m}{n}$ is an upper bound and $\frac{m-1}{n}$ is not upper bound

For both two cases, we can find m. s.t. $L < m \le K$

$\frac{m}{n}$ is an upper bound and $\frac{m-1}{n}$ is not upper bound. #

Ex. 5.5.4.

Proof. ① $|q_n - q_{n'}| \le \frac{1}{M}$, $\forall n, n' \ge M$. $\Rightarrow \{q_n\}$ Cauchy.

Given $\varepsilon > 0$, choose large $M \ge 1$, s.t. $\frac{1}{M} \le \varepsilon$, $\forall i, j \ge M$.

$\Rightarrow |q_i - q_j| \le \frac{1}{M} \le \varepsilon$.

$\Rightarrow \{q_n\}$ is a Cauchy sequence.

② $s = \mathrm{LIM}_{n\to\infty} q_n \Rightarrow |q_M - s| \le \frac{1}{M}$. $\forall M \ge 1$.

Given $M \ge 1$, $\because |q_n - q_{n'}| \le \frac{1}{M}, \forall n, n' \ge M \Rightarrow |q_n - q_M| \le \frac{1}{M}, \forall n \ge M$.

$\Rightarrow q_M - \frac{1}{M} \le q_n \le q_M + \frac{1}{M}, \forall n \ge M. \Rightarrow q_M - \frac{1}{M} \le \mathrm{LIM}_{n\to\infty} q_n = s \le q_M + \frac{1}{M}$

$\Rightarrow |s - q$

Solutions of HW 2.

Ex. 5.6.1. (a) (c) (e) $x, y > 0$ reals, $n, m \geq 1$ positive integers.

Proof: (a) $y = x^{\frac{1}{n}} \Rightarrow y^n = x$.

Need only to show $y^n > x$, $y^n < x$ are not true.

If $y^n > x$. ~~be~~ ~~$\varepsilon = y^n - x > 0$~~

Claim: $\exists\, k \geq 1$, s.t. $(y - \frac{1}{k})^n > x$.

Otherwise, if $(y - \frac{1}{k})^n \leq x$ true for all $k \geq 1$, $\Rightarrow y^n \leq x$. contradiction.

From claim, we know $y - \frac{1}{k}$ is an upper bound of set $\{z \in \mathbb{R}^+ \mid z^n \leq x\}$, but $y = x^{\frac{1}{n}} =$ the least upper bound. Contradiction.

If $y^n < x$.

Claim: $\exists\, k \geq 1$. s.t. $(y + \frac{1}{k})^n < x$.

otherwise If $(y + \frac{1}{k})^n \geq x$, $\forall k \geq 1 \Rightarrow y^n \geq x$ contradiction.

From claim $\Rightarrow y + \frac{1}{k} \in \{z \in \mathbb{R}^+ \mid z^n \leq x\}$.

since $\frac{1}{k} > 0$. $\Rightarrow y$ can't be an upper bound of $\{z \in \mathbb{R}^+ \mid z^n \leq x\}$

Contradiction to $y = x^{\frac{1}{n}}$ the least upper bound. #

(c) $x^{\frac{1}{n}}$ is positive.

Since $\lim_{k \to \infty} \frac{1}{k^n} = 0$. $\Rightarrow \exists\, K > 0$. $\frac{1}{k^n} \leq x$. for $k \geq K$. from $x > 0$.

$\Rightarrow \frac{1}{K} \in \{y \in \mathbb{R} \mid y \geq 0, \& \ y^n \leq x\} \Rightarrow x^{\frac{1}{n}} \geq \frac{1}{K} > 0$.

(e). $x > 1$, $x^{\frac{1}{k}}$ decreasing as k; $0 < x < 1$, $x^{\frac{1}{k}}$ increasing as k; $x = 1$, $x^{\frac{1}{k}} = 1$.

only show $x > 1$, $x^{\frac{1}{k}}$ decreasing. i.e. If $k_1 > k_2$, $x^{\frac{1}{k_1}} < x^{\frac{1}{k_2}}$

Since $(x^{\frac{1}{k_1}})^{k_1} = x$, $(x^{\frac{1}{k_2}})^{k_2} = x$

If $x^{\frac{1}{k_1}} = x^{\frac{1}{k_2}} \Rightarrow (x^{\frac{1}{k_2}})^{k_1 - k_2} = 1 \Rightarrow x^{\frac{1}{k_2}} = 1 \Rightarrow x = 1$ contradiction.

If $x^{\frac{1}{k_1}} > x^{\frac{1}{k_2}} \Rightarrow (x^{\frac{1}{k_2}})^{k_1 - k_2} < 1 \Rightarrow x^{\frac{1}{k_2}} < 1 \Rightarrow x < 1$

(Using (d). $x > y \geq 0$ iff $x^{\frac{1}{n}} > y^{\frac{1}{n}} \geq 0$).

$0 < x < 1$, $x^{\frac{1}{k}}$ increasing, by same argument. #

$n \to \infty$

(3). x, y positive $\Rightarrow x + y$ and xy positive.