Solutions of HW 3.

Ex. 6.1.2. (simply unwrap all the definitions)

Proof: $\{a_n\}_{n=m}^{\infty}$ converges to L

iff $\{a_n\}_{n=m}^{\infty}$ is eventually ε-close to L for every $\varepsilon > 0$

iff For every $\varepsilon > 0$, $\exists N \geq m$, such that $\{a_n\}_{n=N}^{\infty}$ is ε-close to L

iff $\forall \varepsilon > 0$, $\exists N \geq m$, s.t. $|a_n - L| \leq \varepsilon$, $\forall n \geq N$. #

Ex. 6.1.5. (Convergent sequences are Cauchy)

Proof: Let $\{a_n\}_{n=m}^{\infty}$ convergent to L, i.e.

$\forall \varepsilon > 0$, $\exists N \geq m$, s.t. $|a_n - L| \leq \varepsilon$, $\forall n \geq N$.

To check $\{a_n\}_{n=m}^{\infty}$ is Cauchy: need to show

$\forall \varepsilon' > 0$, $\exists N' \geq m$, s.t. $|a_i - a_j| \leq \varepsilon'$, $\forall i, j \geq N'$.

Now take $\varepsilon = \frac{\varepsilon'}{2}$, for $N' = N$, and $\forall i, j \geq N'$

$|a_i - a_j| = |a_i - L + L - a_j| \leq |a_i - L| + |a_j - L| \leq 2\varepsilon = \varepsilon'$. #

Ex. 6.1.8. (Thm 6.1.19 b) e) g)) $x = \lim\limits_{n\to\infty} a_n$, $y = \lim\limits_{x\to\infty} b_n$

Proof: b). $\lim\limits_{n\to\infty}(a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$

$\{a_n\}, \{b_n\}$ convergent $\Rightarrow$ bounded, $\Rightarrow \exists M > 0$, s.t. $|a_n| \leq M$, $|b_n| \leq M$, $\forall n \geq 1$.

Since $x = \lim\limits_{n\to\infty} a_n$, $y = \lim\limits_{n\to\infty} b_n$, For any $\varepsilon > 0$, $\exists N \geq 1$, s.t.

$|a_n - x| \leq \varepsilon$, $|b_n - x| \leq \varepsilon$, $\forall n \geq N$.

Now for any $\varepsilon' > 0$, need to find $N' \geq 1$, s.t.

$|a_n b_n - xy| \leq \varepsilon'$ $\forall n \geq N'$.

Take $\varepsilon = \frac{\varepsilon'}{2M} > 0$, N' be corresponding N w.r.t. ε. for $\{a_n\}$ & $\{b_n\}$.

for $n \geq N'$, $|a_n b_n - xy| = |a_n b_n - a_n y + a_n y - xy| \leq |a_n||b_n - y| + |a_n - x||y|$

$\leq M \cdot \varepsilon + M \cdot \varepsilon = \varepsilon'$. #

e). $y \neq 0$, $b_n \neq 0$. $\forall n \geq 1$, $\Rightarrow \lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}$.

Pf: Need to show: $\forall \varepsilon > 0$, $\exists N \geq 1$, s.t.

$$|b_n^{-1} - y^{-1}| \leq \varepsilon, \quad \forall n \geq N.$$

First since $y \neq 0$, $\lim_{n\to\infty} b_n = y \Rightarrow$ For $\varepsilon_1 = \frac{|y|}{2} > 0$, $\exists N_1 \geq 1$.

$$|b_n - y| \leq \frac{|y|}{2}. \quad \forall n \geq N_1.$$

$\Rightarrow$ For $n \geq N_1$, $|b_n| = |y + b_n - y| \geq |y| - |b_n - y| \geq \frac{|y|}{2} > 0$.

And For $\varepsilon_2 = \frac{\varepsilon |y|^2}{2} > 0$, $\exists N_2 \geq 1$, s.t.

$$|b_n - y| \leq \varepsilon_2, \quad \forall n \geq N_2.$$

Let $N = N_1 + N_2$, for $n \geq N$, we have

$$|b_n^{-1} - y^{-1}| = \frac{|b_n - y|}{|b_n| \cdot |y|} \leq \frac{\varepsilon_2}{|y| \cdot \frac{|y|}{2}} = \frac{\varepsilon |y|^2}{2} \Big/ \frac{|y|^2}{2} = \varepsilon. \quad \#$$

(g). $\lim_{n\to\infty} \max(a_n, b_n) = \max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$

Pf: Case 1. $x = \lim_{n\to\infty} a_n = y = \lim_{n\to\infty} b_n$

This means $\{a_n\}$ & $\{b_n\}$ are equivalent. i.e.

$\forall \varepsilon > 0$, $\exists N \geq 1$, $\Rightarrow |a_n - b_n| \leq \varepsilon$, $\forall n \geq N$. $\quad (*)$

let $c_n = \max(a_n, b_n)$, we need only to show $\{a_n\}$ & $\{c_n\}$ equivalent.

For $\varepsilon > 0$, take $N \geq 1$ as in $(*)$, for any $n \geq N$

$$|c_n - a_n| = \begin{cases} |a_n - a_n| = 0, & \text{for } a_n = c_n \\ |b_n - a_n| \leq \varepsilon, & \text{for } c_n = b_n. \end{cases}$$

i.e. $|c_n - a_n| \leq \varepsilon$ true.

Case 2: $x = \lim_{n\to\infty} a_n > y = \lim_{n\to\infty} b_n$.

Let $\varepsilon = \frac{x-y}{2} > 0$, $\{a_n\}$ & $\{b_n\}$ convergent

$\Rightarrow \exists N \geq 1$, s.t. $|a_n - x| \leq \varepsilon$, $|b_n - y| \leq \varepsilon$, $\forall n \geq N$.

$$\left.\begin{aligned} \Rightarrow a_n &= a_n - x + x \geq x - |a_n - x| \geq x - \frac{x-y}{2} = \frac{x+y}{2} \\ b_n &= b_n - y + y \leq y + |b_n - y| \leq y + \frac{x-y}{2} = \frac{x+y}{2} \end{aligned}\right\} \Rightarrow a_n \geq b_n. \ \forall n \geq N.$$

$\Rightarrow \max(a_n, b_n) = a_n$ for $n \geq N$. $\Rightarrow \lim_{n\to\infty} \max(a_n, b_n) = \lim_{n\to\infty} a_n = x$ #

Case 3: $x = \lim_{n\to\infty} a_n < y = \lim_{n\to\infty} b_n$. same as Case 2. #

Solutions of HW3

Ex 6.1.9. (Why Thm 6.1.19(f) fails when $y=0$).

Solution: One may see a counterexample as:

$a_n = 1$, $b_n = \frac{1}{n}$. $\forall n \geq 1$. $\frac{a_n}{b_n} = n$.

$\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = 0$.

but $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist. #

Ex. 6.2.2. (Proposition 6.2.11. b)).

Proof: M is an upper bound of E in $\mathbb{R}^*$, we have ~~three~~ cases:

Case 1: $M = +\infty$,

Since $\sup(E) \in \mathbb{R}^*$, and we have $+\infty \geq y$, $\forall y \in \mathbb{R}^*$

$\Rightarrow \sup(E) \leq M$ true.

Case 2: $M = -\infty$.

Since M is an upper bound of E.

$\Rightarrow E = \{-\infty\}$ or ϕ empty set. $\Rightarrow \sup(E) = -\infty$.

$\Rightarrow \sup(E) \leq M$ true.

Case 3: $M \in \mathbb{R}$.

Since $\sup(E)$ = the least upper bound of E or $-\infty$.

$\Rightarrow \sup(E) \leq M$ true. #

Ex 6.3.2. (Proposition 6.3.6)

Proof: ① For $x \in \mathbb{R}^*$, $x = \sup_{n\geq m} a_n \Rightarrow a_n \leq x \quad \forall n \geq m$.

Since $x = \sup\{a_n \mid n \geq m\}$, $\Rightarrow x$ = the least upper bound or $+\infty$.

$\Rightarrow a_n \leq x$ true $\forall n \geq m$. #

②. $M \in \mathbb{R}^*$, M is an upper bound of $\{a_n\}_{n=m}^{\infty} \Rightarrow M \geq x$.

Case 1: $M = +\infty$, $\Rightarrow M \geq x$ true any way.

Case 2: $M \in \mathbb{R}$.

(continue) $\Rightarrow$ set $\{a_n \mid n \geq m\}$ has an upper bound

$\Rightarrow$ the least upper bound exists and finite

i.e. $x = \sup_{n \geq m} a_n$ finite and the least upper bound.

$\Rightarrow x \leq M$ true. *

③. For any $y \in \mathbb{R}^*$, $y < x \Rightarrow \exists\, n \geq m$, s.t. $y < a_n \leq x$.

Case 1. $y = -\infty$; statement is true for any a_n.

Case 2. $y \in \mathbb{R}$; $y < x$.

If $x = +\infty$, i.e. $\{a_n\}_{n=m}^{\infty}$ has no upper bound

$\Rightarrow \exists\, n \geq m$, $a_n > y$ true since y finite.

If $x \in \mathbb{R}$, $x = \sup_{n \geq m} a_n$.

Assume $y \geq a_n$, $\forall\, n \geq m$. $\Rightarrow y$ is an upper bound of $\{a_n\}_{n=m}^{\infty}$

~~but~~ x is the least upper bound

$\Rightarrow x \leq y$ Contradict to $x > y$.

Which means $\exists\, n \geq m$. s.t. $y < a_n \leq x$ true. #

Ex. 6.3.4. (Why Proposition 6.3.10 fails for $x > 1$)

Proof: Assume for $x > 1$, $\{x^n\}$ convergent to $L \in \mathbb{R}$.

By Prop 6.3.10, since $0 < \frac{1}{x} < 1$. $\{(\frac{1}{x})^n\}$ converges to 0.

$$\lim_{n\to\infty} x^n \cdot \left(\frac{1}{x}\right)^n = \lim_{n\to\infty} x^n \lim_{n\to\infty} \left(\frac{1}{x}\right)^n = L \cdot 0 = 0.$$

but $$\lim_{n\to\infty} x^n \left(\frac{1}{x}\right)^n = \lim_{n\to\infty} 1 = 1$$

contradiction #.