Solutions of HW4

Ex. 6.4.7: (zero test for sequences).

Pf: ① $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} |a_n| = 0$.

It is from the definition of convergence. Notice that
$|a_n - 0| \le \varepsilon \Leftrightarrow \big|\,|a_n| - 0\big| \le \varepsilon$.

② $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$.

$-|a_n| \le a_n \le |a_n|$, $\forall n \ge 1$. $\lim_{n\to\infty} |a_n| = 0$, & $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$.

By squeeze test $\Rightarrow \lim_{n\to\infty} a_n = 0$ #.

If not converge to 0, the statement is not true.

let $a_n = (-1)^n$. $n \ge 1$, $\{a_n\}_{n=1}^{\infty}$ is divergent. but $\{|a_n|\}_{n=1}^{\infty}$ is convergent. #

Ex. 6.4.8. let $L^+ = \limsup_{n\to\infty} a_n$, $L^- = \liminf_{n\to\infty} a_n$

Pf: L^+ is a limit point of $\{a_n\}_{n=1}^{\infty}$, and is the largest limit point of $\{a_n\}_{n=1}^{\infty}$

Case 1: $L^+ = +\infty \Rightarrow \{a_n\}_{n=M}^{\infty}$ is unbounded. since $L^+ = \inf_{N\ge M} a_N^+ = \inf_{N\ge M} \sup_{n\ge N} \{a_n \mid n \ge N\}$

$\Rightarrow L^+$ is a limit point of $\{a_n\}_{n=1}^{\infty}$

and for any limit point $c \in \mathbb{R}^*$, $c \le +\infty = L^+$ true.

Case 2: $L^+ \in \mathbb{R}$. from proposition 6.4.12 $\Rightarrow$ statement is true.

Case 3. $L^+ = -\infty$.

a). L^+ is a limit point. i.e. $\{a_n\}_{n=1}^{\infty}$ has no lower bound.
if not, assume $\exists\, c \in \mathbb{R}$, s.t. $a_n \ge c$ true $\Rightarrow a_N^+ \ge c$ true $\forall N \ge 1$.
$\Rightarrow L^+ \ge c$ contradiction

b). L^+ is the largest limit point. if not, assume $\exists\, c \in \mathbb{R}$, s.t.
c is a limit point of $\{a_n\}_{n=1}^{\infty}$.
$\Rightarrow$ ~~a subsequence~~ $\forall N \ge 1$, $\exists\, n_N \ge N$, s.t. $|a_{n_N} - c| \le 1$, $\Rightarrow a_{n_N} \ge c - 1$.
$\Rightarrow a_N^+ = \sup_{n\ge N} a_n \ge c - 1$, $L^+ = \inf_{N\ge 1} a_N^+ \ge c - 1$.
Contradiction. #. Same argument for L^-. #

Solutions of HW4.

Ex. 6.4.10: let $\{b_m\}_{m=1}^{\infty}$ are sequence of limit points of $\{a_n\}_{n=1}^{\infty}$. x is a limit point of $\{b_m\}$.

Proof: Need to show x is a limit point of $\{a_n\}_{n=1}^{\infty}$.

Need to show: $\forall\ \varepsilon > 0,\ N \geq 1,\quad \exists\ n_{N,\varepsilon} \geq N$.

s.t. $|a_{n_{N,\varepsilon}} - x| < \varepsilon$.

First. since x is a limit point of $\{b_m\}_{m=1}^{\infty}$.

for $\varepsilon' = \frac{\varepsilon}{2} > 0$, $\exists\ m_{\varepsilon'} \geq 1$. s.t. $|b_{m_{\varepsilon'}} - x| \leq \varepsilon'$.

And since $b_{m_{\varepsilon'}}$ is a limit point of $\{a_n\}_{n=1}^{\infty}$.

For $\varepsilon' = \frac{\varepsilon}{2} > 0$. $N \geq 1$. $\exists\ n_{N,\varepsilon'} \geq N$.

s.t. $|a_{n_{N,\varepsilon'}} - b_{m_{\varepsilon'}}| \leq \varepsilon'$.

$$\Rightarrow |a_{n_{N,\varepsilon'}} - x| = |a_{n_{N,\varepsilon'}} - b_{m_{\varepsilon'}} + b_{m_{\varepsilon'}} - x|$$

$$\leq |a_{n_{N,\varepsilon'}} - b_{m_{\varepsilon'}}| + |b_{m_{\varepsilon'}} - x| \leq \varepsilon' + \varepsilon' = \varepsilon. \quad \#$$

Ex 6.5.1: Show $\lim_{n\to\infty} \frac{1}{n^q} = 0$. $\forall$ rational $q > 0$, & $\lim_{n\to\infty} n^q$ not exist.

pf: let $q = \frac{a}{b}$. for integer $a, b \geq 1$.

$$\frac{1}{n^q} = \left(\left(\frac{1}{n}\right)^{\frac{1}{b}}\right)^a = \left(\frac{1}{n^{\frac{1}{b}}}\right)^a$$

Since $\lim_{n\to\infty} \frac{1}{n^{\frac{1}{b}}} = 0$, $\Rightarrow \lim_{n\to\infty} \frac{1}{n^q} = \overbrace{\lim_{n\to\infty} \frac{1}{n^{\frac{1}{b}}} \cdots \lim_{n\to\infty} \frac{1}{n^{\frac{1}{b}}}}^{a \text{ copies}} = 0 \cdots 0 = 0$ #

If $\lim_{n\to\infty} n^q = L$, $1 = \lim_{n\to\infty}\left(n^q \cdot \frac{1}{n^q}\right) = \lim_{n\to\infty} n^q \cdot \lim_{n\to\infty} \frac{1}{n^q} = L \cdot 0 = 0$ Contradiction #

Ex. 6.6.2. Find two different sequences $\{a_n\}_{n=1}^{\infty}$ & $\{b_n\}_{n=1}^{\infty}$, they are subsequences of each others.

Solution: $\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ $\quad \{b_n\}_{n=1}^{\infty} = \{(-1)^{n+1}\}_{n=1}^{\infty}$. #

Ex. 6.6.3: $\{a_n\}_{n=1}^{\infty}$ unbounded.

pf: Since $\{a_n\}_{n=1}^{\infty}$ unbounded, $\Rightarrow$

$\forall M > 0$, $\exists\ n_M \geq 1$. s.t. $|a_{n_M}| \geq M$. true.

Solutions of HW4.

Let $M = j$. ~~for~~ $j \in \mathbb{N}$, pick $n_j \geq 1 + n_{j-1}$

s.t. $|a_{n_j}| \geq j$.

(If not, $\exists\, j \in \mathbb{N}$. all k, s.t. $|a_k| \geq j$, satisfies $k \leq n_{j-1}$

which imply $|a_n| \leq j$ $\forall\, j \geq n_{j-1}$. but $\{a_n\}_{n=1}^{n_{j-1}}$ finite $\Rightarrow$ bounded too.

$\Rightarrow \{a_n\}$ bounded, contradiction)

let $b_j = a_{n_j}$. $\quad |b_j| \geq j \Rightarrow \frac{1}{|b_j|} \leq \frac{1}{j}$

$\Rightarrow \lim_{j\to\infty} \frac{1}{b_j} = 0$. by squeeze test. #

Ex. 6.6.5 (limit point $\Leftrightarrow$ subsequence conv.).

Pf: (a) $\Rightarrow$ (b) let L be a limit point of $\{a_n\}_{n=1}^{\infty}$.

$\Leftrightarrow \forall\, \varepsilon > 0$. $N \geq 1$, $\exists\, n_N \geq N$, s.t. $|a_{n_N} - L| \leq \varepsilon$. $\quad (*)$

Now define subsequence $\{a_{n_j}\}$ inductively:

$n_1 = \min\{n \in \mathbb{N} \mid |a_n - L| \leq 1\}$.

$\vdots$

$n_j = \min\{n \geq n_{j-1}+1 \mid |a_n - L| \leq \frac{1}{j}\}$, for $j \geq 2$.

$\vdots$

From $(*)$ choose $\varepsilon = \frac{1}{j}$, $N = n_{j-1}+1$, $\Rightarrow n_j$ exists. and $n_j > n_{j-1}$.

$\Rightarrow \{a_{n_j}\}_{j=1}^{\infty}$ is a subseq of $\{a_n\}_{n=1}^{\infty}$

and $\lim_{j\to\infty} a_{n_j} = \lim_{j\to\infty}(a_{n_j} - L) + L = L$. since $\lim_{j\to\infty}|a_{n_j} - L| = 0$.

$\Rightarrow \{a_{n_j}\}_{j=1}^{\infty}$ converges to L. #

(b) $\Rightarrow$ (a). Assume subseq $\{a_{n_j}\}_{j=1}^{\infty}$ converges to L.

$\Leftrightarrow \forall\, \varepsilon > 0$, $\exists\, J \geq 1$, s.t. $|a_{n_j} - L| \leq \varepsilon$, $\forall\, j \geq J$.

Now need to show L is a limit point.

For $\forall\, \varepsilon > 0$, $N \geq 1$, pick $j \geq J$. s.t. $n_j \geq n_J \geq N$.

$\Rightarrow |a_{n_j} - L| \leq \varepsilon$.

$\Rightarrow L$ is a limit point of $\{a_n\}_{n=1}^{\infty}$ #