# Solutions of HW5.



Ex 7.1.1. Pf: (e) using math induction on $n$.

(f). using math induction on $n$.

Ex. 7.1.4. Show $(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$

Pf: ① $n=1$, $LS = x+y$, $RL = \sum_{j=0}^{1} \frac{1!}{j!(1-j)!} x^j y^{n-j} = x+y$

$\Rightarrow$ statement is true.

② Assume $n=k$, statement is true.

For $n=k+1$,

$$LS = (x+y)^{k+1} = (x+y)^k (x+y) = \left(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k-j}\right)(x+y)$$

$$= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^{j+1} y^{k-j} + \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k+1-j}$$

$$= \sum_{j=1}^{k+1} \frac{k!}{(j-1)!(k+1-j)!} x^j y^{k+1-j} + \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k+1-j}.$$

Since $\frac{k!}{(j-1)!(k+1-j)!} + \frac{k!}{j!(k-j)!} = \frac{(k+1)!}{j!(k+1-j)!}$ for $j=1,2,\dots,k$.

and $\frac{k!}{(k+1-1)!(k+1-(k+1))!} = \frac{(k+1)!}{(k+1)!(k+1-(k+1))!}$, $\frac{k!}{0!(k-0)!} = \frac{(k+1)!}{0!(k+1-0)!}$

$$\Rightarrow LS = \sum_{j=0}^{k+1} \frac{(k+1)!}{j!(k+1-j)!} x^j y^{k+1-j} = RS.$$

By math Induction, statement is true for all $n \geq 1$.

Ex. 7.1.5 Pf: Induct on $\#(X) = k$. $\lim_{n\to\infty} \sum_{x\in X} a_n(x) = \sum_{x\in X} \lim_{n\to\infty} a_n(x)$

① $\#(X)=0$, $LS=0$, $RS=0$ statement true

② Assume $\#(X) = k$, true. For $\#(X) = k+1$. Let $x_{k+1} \in X$.

$$LS = \lim_{n\to\infty} \sum_{x\in X} a_n(x) = \lim_{n\to\infty} \sum_{x\in X\setminus\{x_{k+1}\}} a_n(x) + \lim_{n\to\infty} a_n(x_{k+1}).$$

$$= \sum_{x\in X\setminus\{x_{k+1}\}} \lim_{n\to\infty} a_n(x) + \lim_{n\to\infty} a_n(x_{k+1}) = \sum_{x\in X} \lim_{n\to\infty} a_n(x) = RS.$$

Continue. ②

(Here we use induction assume $\lim_{n\to\infty} \sum_{x\in X\setminus\{x_{k+1}\}} a_n(x) = \sum_{x\in X\setminus\{x_{k+1}\}} (\lim_{n\to\infty} a_n(x))$ since $\#(X\setminus\{x_{k+1}\}) = k$.)

Hence, by math induction $\Rightarrow$ Statement is true for all $k = \#(X)$ #

Ex. 7.2.2. pf: $\sum_{n=0}^{\infty} a_n$ convergent iff the partial sums $\{S_N\}$ convergent.

$\Leftrightarrow \forall \varepsilon > 0, \exists N \geq m$, s.t. $|S_q - S_{p-1}| \leq \varepsilon$, $\forall q, p \geq N$.

Since $S_q - S_{p-1} = \sum_{n=p}^{q} a_n$.

$\Leftrightarrow \forall \varepsilon > 0, \exists N \geq m$, s.t. $\left|\sum_{n=p}^{q} a_n\right| \leq \varepsilon$, $\forall p, q \geq N$. #

Ex. 7.2.6. pf: For series $\sum_{n=0}^{\infty} (a_n - a_{n+1})$.

$S_N = \sum_{n=0}^{N} (a_n - a_{n+1}) = a_0 - a_{N+1}$. by math Induction

Since $\lim_{n\to\infty} a_n = 0$. $\Rightarrow \lim_{N\to\infty} a_{N+1} = 0$

$\Rightarrow \lim_{N\to\infty} S_N = \lim_{N\to\infty} (a_0 - a_{N+1}) = a_0 - \lim_{N\to\infty} a_{N+1} = a_0$.

$\Rightarrow \sum_{n=0}^{\infty} (a_n - a_{n+1}) = a_0$. #