Solutions of HW6.

Ex. 7.3.1. Pf: First show $\sum_{n=m}^{\infty}|a_n| \le \sum_{n=m}^{\infty} b_n$.

$|a_n| \le b_n \ \forall n \Rightarrow \sum_{n=m}^{N}|a_n| \le \sum_{n=m}^{N} b_n \le \sum_{n=m}^{\infty} b_n$

by Prop. 7.3.1 $\Rightarrow \sum_{n=m}^{\infty}|a_n|$ convergent.

From Prop. 7.2.9 $\Rightarrow \left|\sum_{n=m}^{\infty} a_n\right| \le \sum_{n=m}^{\infty}|a_n|$ true. #

Ex. 7.3.2 Pf: For $|x| \ge 1$. since $\lim_{n\to\infty} x^n \ne 0$. by zero test. $\sum_{n=0}^{\infty} x^n$ divergent.

For $|x|<1$. for $\sum_{n=0}^{\infty} x^n$.

$S_N = \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$ by math induction.

Since $\lim_{N\to\infty} |x|^{N+1} = 0 \Rightarrow \lim_{N\to\infty} x^{N+1} = 0$.

$\Rightarrow \lim_{N\to\infty} S_N = \frac{1}{1-x} - \frac{1}{1-x}\lim_{N\to\infty} x^{N+1} = \frac{1}{1-x}$.

$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ #.

Ex. 7.3.3 Pf: $\sum_{n=0}^{\infty}|a_n| = 0$. $\sum_{n=0}^{\infty} a_n$ absolutely convergent.

$\Rightarrow S_N = \sum_{n=0}^{N}|a_n|$ convergent to 0.

and $\{S_N\}_{N=0}^{\infty}$ increasing seq.

$\Rightarrow S_N = 0. \quad \forall N \ge 0$.

$\Rightarrow |a_{N+1}| = S_{N+1} - S_N = 0 - 0 = 0 \Rightarrow a_n = 0 \ \forall n \ge 0$.

Ex. 7.4.1 Pf: Let $\{S_N\}$ be partial sums of $\sum_{n=0}^{\infty}|a_n|$, $\{T_M\}$ be the partial sums of $\sum_{m=0}^{\infty}|a_{f(m)}|$

Since $f: N \to N$ increasing, $\Rightarrow f$ is injective.

$\Rightarrow f(\{0,1,\dots,M\}) \subseteq \{0,1,\dots,f(M)\}$ for $\forall M \ge 0$.

$\Rightarrow T_M = \sum_{m=0}^{M}|a_{f(m)}| \le \sum_{n=0}^{f(M)}|a_n| = S_{f(M)} \le \sum_{n=0}^{\infty}|a_n|$

$\Rightarrow \sum_{m=0}^{\infty}|a_{f(m)}|$ convergent. #

(Continue) ②

Ex 7.5.2 : pf :
$\sum_{n=1}^{\infty} n^q x^n$ for $|x|<1$. q real.

using ratio test. $\lim_{n\to\infty} \frac{|(n+1)^q x^{n+1}|}{|n^q x^n|} = \lim_{n\to\infty} (1+\frac{1}{n})^q |x| = |x| < 1$.

$\Rightarrow \sum_{n=1}^{\infty} n^q x^n$ absolutely convergent.

by ~~zero~~ test $\Rightarrow \lim_{n\to\infty} n^q x^n = 0$. #

Ex. 7.5.3 pf : Example 1 : $\sum_{n=1}^{\infty} \frac{1}{n}$. divergent series.

and $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$, $\lim_{n\to\infty} (\frac{1}{n})^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$

Example 2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent series.

and $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} (\frac{n}{n+1})^2 = 1$,

$\lim_{n\to\infty} (\frac{1}{n^2})^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^{\frac{1}{n}}}\right)^2 = 1$.