Solutions of HW 7

Ex 8.1.4: $f: \mathbb{N} \to Y$ function. $f(\mathbb{N})$ at most countable.

Pf. Define $A = \{n \in \mathbb{N} \mid f(m) \neq f(n) \ \forall\, 0 \le m < n\}$.

claim: $f: A \to f(\mathbb{N})$ is a bijection.

① f is one to one.

$\forall\, n_1 > n_2 \ge 0$, $n_1, n_2 \in A$, by def of A, $\Rightarrow f(n_1) \neq f(m)$. $\forall\, 0 \le m < n_1$

$\Rightarrow f(n_1) \neq f(n_2)$

② f is onto.

$\forall\, y \in f(\mathbb{N})$ i.e. $\exists\, n \in \mathbb{N}$. $y = f(n)$, let $n_0 = \min\{m \in \mathbb{N} \mid f(m) = f(n)\}$

$\Rightarrow n_0 \in A$ since $f(n_0) \neq f(m)$ $\forall\, 0 \le m < n_0$.

$\Rightarrow f(n_0) = f(n) = y$.

Since $A \subseteq \mathbb{N}$ is at most countable

$\Rightarrow f(\mathbb{N})$ is at most countable #.

EX 8.1.6: A at most countable iff $\exists$ a injective map $f: A \to \mathbb{N}$.

Pf: "$\Rightarrow$" A at most countable $\Rightarrow$ A finite or countable.

① A ~~finite~~. let $A = \{x_0, x_1, \cdots, x_n\}$

Define $f: A \to \mathbb{N}$ as $f(x_i) = i$ $0 \le i \le n$, f is injective.

② A countable $\Rightarrow \exists$ bijection $f: A \to \mathbb{N}$. $\Rightarrow f$ in injective.

"$\Leftarrow$" $f: A \to \mathbb{N}$ injective.

$\Rightarrow f: A \to f(A)$ is surjective $\Rightarrow f: A \to f(A)$ bijective

Since $f(A) \subseteq \mathbb{N}$ at most countable $\Rightarrow$ A is at most countable #

Ex 8.1.7: X, Y countable $\Rightarrow X \cup Y$ countable.

Pf: X, Y countable $\Rightarrow \exists$ bijection $f: \mathbb{N} \to X$, $g: \mathbb{N} \to Y$.

Define $h(2n) = f(n)$, $h(2n+1) = g(n)$ $\forall\, n \ge 0$.

$h: \mathbb{N} \to X \cup Y$ is surjective. i.e. $h(\mathbb{N}) = X \cup Y$.

$\Rightarrow X \cup Y$ countable by ex. 8.1.4. #

Continue

Ex.8.1.9. I at most countable, X_α at most countable for each $\alpha \in I$, $\Rightarrow \bigcup_{\alpha\in I} X_\alpha$ at most countable.

Pf: I at most countable $\Rightarrow \exists$ surjection $f : \mathbb{N} \to I$.

X_α at most countable $\Rightarrow \exists$ surjection $g_{f(n)} : \mathbb{N} \to X_\alpha$.

Define $g : \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha\in I} X_\alpha$ as

$g(n, m) = g_{f(n)}(m) \quad \Rightarrow g$ is a surjection.

since $\mathbb{N}\times\mathbb{N}$ is countable. $\Rightarrow \exists\ \bar{g} : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ bijection

Define $h = g\circ\bar{g} : \mathbb{N} \to \bigcup_{\alpha\in I} X_\alpha$.

h is a surjection, i.e. $h(\mathbb{N}) = \bigcup_{\alpha\in I} X_\alpha$.

$\Rightarrow \bigcup_{\alpha\in I} X_\alpha = h(\mathbb{N})$ is at most countable from ex. 8.1.4. #

Ex 8.2.2 $\sum_{x\in X} f(x)$ absolutely convergent $\Rightarrow \{x\in X \mid f(x)\neq 0\}$ at most countable.

Pf: let $M = \sup\{\sum_{x\in A}|f(x)| : A \subseteq X,\ A \text{ finite subset}\} < +\infty$

Define $A_n = \{x\in X \mid |f(x)| > \frac{1}{n}\}$ for $n = 1, 2, 3, \ldots$

claim: A_n finite, for each $n \geq 1$.

If not, $\exists\ n_0 \geq N$, s.t. A_{n_0} is an infinite set.

$\Rightarrow \exists$ a subset B of A_{n_0} with $\#(B) > n_0 \cdot M$

Since $M \geq \sum_{x\in B}|f(x)| > \frac{1}{n_0}\cdot\#(B) > \frac{1}{n_0}\cdot n_0 M = M$ contradiction.

Now $\{x\in X \mid f(x)\neq 0\} = \bigcup_{n=1}^{\infty} A_n$ since A_n finite for each $n\geq 1$

from Ex 8.1.9 $\Rightarrow \{x\in X \mid f(x)\neq 0\}$ is at most countable #.

Ex 8.2.6. $\sum_{n=0}^{\infty} a_n$ conditionally conv. but not absolutely conv.; show there exists a bijection $f : \mathbb{N} \to \mathbb{N}$, s.t. $\sum_{m=0}^{\infty} a_{f(m)}$ diverges to $+\infty$.

Pf: let $A_+ = \{n\in\mathbb{N} \mid a_n \geq 0\} = \{n_1 < n_2 < n_3 < \cdots\}$

$A_- = \{n\in\mathbb{N} \mid a_n < 0\} = \{m_1 < m_2 < m_3 < \cdots\}$.

since $\sum_{n=0}^{\infty} a_n$ conv. $\Rightarrow \lim_{n\to\infty} a_n = 0$. $\Rightarrow \exists M \geq 0$. s.t. $|a_n| \leq M \quad \forall n\geq 0$.

Continue

We know that

$\sum_{n \in A_+} a_n = a_{n_1} + a_{n_2} + a_{n_3} + \cdots = +\infty$

$\sum_{m \in A_-} a_m = a_{m_1} + a_{m_2} + a_{m_3} + \cdots = -\infty$.

Define bijection $f: \mathbb{N} \to \mathbb{N}$ inductively as

Step 1. pick $f(0), f(1), \ldots, f(k_1)$ from A_+ in order

s.t. $a_{f(0)} + a_{f(1)} + \cdots + a_{f(k_1)} \geq 1$

Step 1.1. pick $f(k_1+1) = m_1$ from A_-. since $|a_n| \leq M \ \forall n \geq 0$

$\Rightarrow a_{f(0)} + a_{f(1)} + \cdots + a_{f(k_1)} + a_{f(k_1+1)} \geq 1 - M$.

⋮

Step k. pick $f(k_{i-1}+2), f(k_{i-1}+3), \ldots, f(k_i)$ from A_+ in order, and at least one.

s.t. $a_{f(0)} + \cdots + a_{f(k_i)} \geq i$

Step i.1. pick $f(k_i+1) = m_i$.

$\Rightarrow a_{f(0)} + \cdots + a_{f(k_i)} + a_{f(k_i+1)} \geq i - M$.

Check f is a bijection.

① f is one to one, from definition.

② f is onto. since $\forall n \in \mathbb{N}$, a_n was picked before step k+1 or step k+1.1

$\Rightarrow \exists k \in \mathbb{N}$ s.t. $a_{f(k)} = a_n$ i.e. $f(k) = n$.

Next to show $\sum_{m=0}^{\infty} a_{f(m)}$ diverges to $+\infty$.

let $S_M = \sum_{m=0}^{M} a_{f(m)}$.

claim: $S_m \geq i - M$, $\forall k_i + 1 \leq m \leq k_{i+1}$.

it follows from construction of f (step k+1).

since $f(k_i+2), \ldots, f(k_{i+1}) \in A_+$ which is nonnegative part.

$\Rightarrow S_m \geq i - M \quad \forall m \geq k_i + 1$.

$\Rightarrow \lim_{m \to \infty} S_m = +\infty \Rightarrow \sum_{m=0}^{\infty} a_{f(m)} = +\infty$.

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