Solutions of HW 8

Ex 8.3.2: $A \subseteq B \subseteq C$, $f: C \to A$ bijection. $D_0 = B \setminus A$, $D_{n+1} = f(D_n)$.

① Sets $D_0, D_1, D_2, \ldots$ are disjoint from each other.

Pf: Suppose $D_n \cap D_m \neq \phi$ for some $n > m \geq 0$.

i.e. $\exists\ a \in D_n \cap D_m$, let $b = f^{-m}(a) \in D_0 = B \setminus A$; $c = f^{-n}(a) \in D_0 = B \setminus A$

$\Rightarrow a = f^m(b) = f^n(c)$.

$\Rightarrow b = f^{n-m}(c) \in A$, but $b \in B \setminus A$ contradiction.

② Define $g(x) = f^{-1}(x)$ for $x \in \bigcup_{n=1}^{\infty} D_n = D$, $g(x) = x$ for $x \in A \setminus D$.

$\Rightarrow g: A \to B$ is a bijection.

Pf. ① g is well-defined since $D \subseteq A$ and f is bijection.

② g is one to one.

Given any $x \neq y \in A = D \cup (A \setminus D)$. Three cases: $x, y \in D$; $x, y \in A \setminus D$; $x \in D$, $y \in A \setminus D$.

If $x, y \in D$, $\Rightarrow g(x) = f^{-1}(x)$, $g(y) = f^{-1}(y)$, f bijection $\Rightarrow g(x) \neq g(y)$.

If $x, y \in A \setminus D \Rightarrow g(x) = x$, $g(y) = y$. $\Rightarrow g(x) \neq g(y)$

If $x \in D$, $y \in A \setminus D \Rightarrow g(x) = x \in A$, $g(y) = f^{-1}(y) \in B \setminus A \Rightarrow g(x) \neq g(y)$.

③ g is onto. $\forall\ y \in B = (B \setminus A) \cup D \cup (A \setminus D)$

If $y \in A \setminus D$, since $g(y) = y$ for $y \in A \setminus D$.

If $y \in (B \setminus A) \cup D$, take $x = f(y)$. $\Rightarrow g(x) = f^{-1}(x) = y$.

Hence $g: A \to B$ is a bijection. #.

Ex. 8.3.4: No power set (i.e. 2^X for set X) can be countably infinite.

Pf: Case 1. X is finite. $\Rightarrow 2^X$ is also finite.

Case 2. X is countable. by Cantor's Theorem

2^X and X have no equal cardinality. $\Rightarrow 2^X$ uncountable.

Case 3. X is uncountable since $Card(2^X) \geq Card(X)$

$\Rightarrow 2^X$ is uncountable too. #

Continue.

Ex. 8.4.2: Statement: disjoint nonempty sets X_α, $\alpha \in I$, $\Rightarrow \exists$ set Y s.t. $\#(Y \cap X_\alpha) = 1$. $\forall \alpha \in I$.

"Axiom of Choice" $\Rightarrow$ "statement".

Pf: From Axiom of choice $\Rightarrow \prod_{\alpha \in I} X_\alpha \neq \phi$. i.e. $\exists \{x_\alpha\}_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$.

Let $Y = \{x_\alpha \mid \alpha \in I\}$, Claim: $Y \cap X_\alpha = \{x_\alpha\}$.

(1) $\{x_\alpha\} \subseteq Y \cap X_\alpha$. true.

(2) $Y \cap X_\alpha \subseteq \{x_\alpha\}$. if not, $\exists x_\beta (\neq x_\alpha) \subseteq Y \cap X_\alpha$.

by definition of Y, $\exists X_\beta$, s.t. $x_\beta \in X_\beta$. $\Rightarrow X_\beta \cap X_\alpha \neq \phi$. contradiction.

"statement" $\Rightarrow$ "Axiom of Choice".

For nonempty sets X_α, $\alpha \in I$. Define new sets $Y_\alpha = \{(\alpha, x) \mid x \in X_\alpha\}$.

sets Y_α, $\alpha \in I$ are disjoint each other. From statement

$\Rightarrow \exists$ set Y, s.t. $\#(Y \cap Y_\alpha) = 1$. i.e. let $Y \cap Y_\alpha = (\alpha, x_\alpha)$. $\forall \alpha \in I$.

$\Rightarrow \{x_\alpha\}_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$. i.e. $\prod_{\alpha \in I} X_\alpha \neq \phi$. #

Ex 8.5.3: $(\mathbb{N} \setminus \{0\}, \mid)$ is partially ordered set.

Pf. Need to show:

a). (Reflexivity). $\forall n \in \mathbb{N} \setminus \{0\}$, $n \mid n$ is true.

b). (Anti-symmetry) $\forall n, m \in \mathbb{N} \setminus \{0\}$. if $n \mid m$. $m \mid n$.

$\Rightarrow m = k_1 n$. $n = k_2 m$. for some $k_1, k_2 \in \mathbb{N} \setminus \{0\}$

$\Rightarrow k_1 = k_2 = 1$. $\Rightarrow n = m$.

c) (Transitivity). For $n, m, k \in \mathbb{N} \setminus \{0\}$. If $n \mid m$, $m \mid k$,

$\Rightarrow \exists k_1, k_2 \in \mathbb{N} \setminus \{0\}$, s.t. $m = k_1 n$, $k = k_2 m$. $\Rightarrow k = k_1 k_2 n$

$\Rightarrow n \mid k$. #

Ex 8.5.5: $f: X \to Y$, Y partially ordered by $\leq_Y$; Define $\leq_X$ by $x \leq_X x'$ iff $f(x) \leq_Y f(x')$.

show $(X, \leq_X)$ partially ordered. If f is one to one.

Pf: need to show:

Continue ③

① (Reflexivity). $\forall x \in X$, since $f(x) \leq_Y f(x)$ true $\Rightarrow x \leq_X x$. true.

② (Anti-symmetry) For $x, y \in X$, if $x \leq_X y$, $y \leq_X x$.
i.e. $f(x) \leq_Y f(y)$, and $f(y) \leq_Y f(x) \Rightarrow f(x) = f(y)$
by f one to one $\Rightarrow x = y$.

③ (Transitivity). For $x, y, z \in X$. If $x \leq_X y$, $y \leq_X z$.
i.e. $f(x) \leq_Y f(y)$, $f(y) \leq_Y f(z)$. $\Rightarrow f(x) \leq_Y f(z)$.
$\Rightarrow x \leq_X z$.

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Ex 8.5.11: X partially ordered; $Y, Y' \subseteq X$ well ordered, show $Y \cup Y'$ well-ordered iff $Y \cup Y'$ totally ordered

Pf: "$\Rightarrow$" If $Y \cup Y'$ well ordered
by def of well ordered $\Rightarrow Y \cup Y'$ totally ordered.

"$\Leftarrow$" If $Y \cup Y'$ totally ordered.
To check $Y \cup Y'$ well ordered. i.e. $\forall A \subseteq Y \cup Y'$,
need to find $\min(A)$ exists.

① If $A \cap Y = \phi$ or $A \cap Y' = \phi$. i.e. $A \subseteq Y'$ or $A \subseteq Y$
then from Y, Y' well ordered $\Rightarrow \min(A)$ exists.

② If $A \cap Y \neq \phi$, $A \cap Y' \neq \phi$. From Y, Y' well ordered
$\Rightarrow \min(A \cap Y)$, $\min(A \cap Y')$ exist.
Define $\min(A) = \min\{\min(A \cap Y), \min(A \cap Y')\}$.
Since $Y \cup Y'$ totally ordered, set $\{\min(A \cap Y), \min(A \cap Y')\}$ totally ordered and well ordered.

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