I. Abstract Duality

(1) Duality in terms of closures. (Assume all sets are finite.)
   (a) Given a matroid $M$, for all $S \subseteq E$ and $x \in E \setminus S$, define $T = E \setminus S \setminus x$. Then either $x \in \text{cl}(S)$ or $x \in \text{cl}^*(T)$, but not both. (This is Oxley’s Exercise 2.2.5.)
   (b) Given two closure operators on a set $E$, such that for all $S \subseteq E$ and $x \in E \setminus S$, either $x \in \text{cl}(S)$ or $x \in \text{cl}^*(E \setminus S \setminus x)$, but not both. Then cl and cl* are the closure operators of a dual pair of matroids. (This is due to Crapo. It’s bigger than (a) because we don’t assume the closures are matroid closures.)

(2) Find a self-dual axiom system based on circuits and cocircuits and the property $|C \cap C^*| \neq 1 \forall C \in \mathcal{C}, C^* \in \mathcal{C}^*.$

II. Duality of Vector Representations

Theorems of concern here are:

• Theorem 2.2.A,B = Exercise 2.2.6(a,b).

• Theorem 2.2.8.

• Theorem 2.2.W (Whitney’s Orthogonality Theorem). In the Euclidean vector space $\mathbb{R}^n$ (with dot product), let $b_1, \ldots, b_n$ be a basis and let $W$ be a subspace. Let $y_i$ be the orthogonal projection of $b_i$ onto $W$ and let $z_i$ be its orthogonal projection onto $W^\perp$. Let $M$ be the vector matroid of $y_1, \ldots, y_n$. Then the vector matroid of $z_1, \ldots, z_n$ is $M^*.$

Problems:

(1) A theorem of linear algebra states: Suppose you have a subspace $W \leq \mathbb{R}^n$ with a basis $\alpha_1, \ldots, \alpha_r$ and you form the matrix $A$ whose rows are the vectors $\alpha_i$, $i = 1, \ldots, r$. Then the orthogonal projection onto $W$ of any vector $x \in \mathbb{R}^n$ is given by the formula $\text{proj}_W x = A^T(AA^T)^{-1}Ax$. Find out how to prove this formula, either by looking it up or by working it out yourself.

(2) Make the assumptions of Theorem 2.2.W with the addition that $\{b_1, b_2, \ldots, b_n\}$ is the standard basis of $\mathbb{R}^n$. Let $A = [y_1, \ldots, y_n]$ and $A^* = [z_1, \ldots, z_n]$ be $n \times n$ matrices. Show directly (using the standard coordinates) that $\mathcal{R}(A)^\perp = \mathcal{R}(A^*)$.

(3) In the situation of Problem (2), how are $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, $W$, and $W^\perp$ related?

III. Transversal Matroids

Remember the bicircular matroid of a graph, $BG(G)$?

(1) Show that $BG(G)$ is a transversal matroid by finding a natural transversal presentation. Which transversal presentations naturally give bicircular matroids?

(2) Characterize the dual bicircular matroids, using the theory of § 2.4 for dual transversal matroids.

IV. Spikes

The graph $2C_n$ is a circle $C_n = e_1e_2 \cdots e_n$ with each $e_i$ doubled by a parallel edge $f_i$. (When needed, I name the vertices $v_1, v_2, \ldots, v_n = v_0$ with $V(e_i) = V(f_i) = \{v_{i-1}, v_i\}$.)

We saw in class that a tippy spike is the extended (or complete) lift matroid $L_0(2C_n, B)$ of a biased graph $(2C_n, B)$. Here $E(L_0) = E(2C_n) \cup \{d_0\}$. The tipless spike is the lift matroid, $L = L_0 \setminus d_0$. I suggest you use that representation of spikes in the following.

(1) Let $I \subseteq [n]$ and define

$$
\begin{align*}
    s_I(e_i) &= f_i, & s_I(f_i) &= e_i & \text{if } i \in I, \\
    s_I(e_i) &= e_i, & s_I(f_i) &= f_i & \text{if } i \notin I.
\end{align*}
$$

Question: When is $s_I$ an isomorphism of $L$ with its dual, $L^\perp$? For instance, we know it is for $I = \emptyset$ when $B = \emptyset$. The answer obviously depends on $B$. 