Construction of Line-Consistent Signed Graphs

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Abstract. The line graph of an edge-signed graph carries vertex signs. A vertex-signed graph is *consistent* if every circle (cycle, circuit) has positive vertex-sign product. Acharya, Acharya, and Sinha recently characterized edge-signed graphs whose line graphs are consistent. Their proof applies Hoede's relatively difficult characterization of consistent vertex-signed graphs. We give a different, constructive proof that does not depend on Hoede's theorem.

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1. Introduction

In the first article on signed graphs—graphs whose edges are labelled positive or negative—Frank Harary [5] gave a simple characterization of those in which the product of signs around every circle (i.e., circuit, cycle) is positive. (Such graphs are called *balanced*.) Later, Beineke and Harary [3] introduced signed vertices and asked the analogous question of characterizing the vertex-signed graphs in which the product of vertex signs around every circle is positive. (These graphs are called *consistent*.) This was more difficult. After preliminary results by Acharya [1] and Rao [7], Hoede found a definitive though complicated answer [6].

An obvious question was never answered until recently. If a signed graph Σ has underlying graph G and edge signature σ, then the line graph $L(G)$ has σ as a vertex signature. Under what conditions is this vertex signature consistent? I call such a graph *line consistent*, as the defining property is consistency of the line graph. Acharya, Acharya, and Sinha [2] found a simple necessary and sufficient condition for line consistency. The necessity of their condition is easy to determine. Sufficiency is not so easy; it depends on Hoede's complicated consistency criterion. Here I give a different proof of sufficiency by showing how to build up, step by step, every signed graph Σ that satisfies Acharya, Acharya, and Sinha's conditions and demonstrating that the construction process preserves consistency of the line graph.

2. Background

All graphs are simple. G denotes a graph, $L(G)$ its line graph. A vertex of degree > 3 is called *supervalent*. A *trail* in G is a sequence $p_0 f_1 p_1 f_2 p_2 \cdots f_l p_l$, alternating between vertices and edges of G, in which $f_i = p_{i-1} p_i$ and the edges are distinct. I employ two notations for a path or circle. In G I use the vertex-edge notation of a trail, with $p_l = p_0$ for a circle (i.e., a closed path). In the line graph the more appropriate notation omits the edges of $L(G)$, thus $f_1 f_2 \cdots f_l$ is a path and $f_1 f_2 \cdots f_l f_1$ is a circle.

A signed graph is written $\Sigma = (G, \sigma)$. The sign of a trail P, written $\sigma(P)$, is the product of its edge signs. A vertex is *totally positive* (or, *totally negative*) if all incident edges are positive (or, negative).

A graph with edge or vertex signs, or both, is *balanced* if every circle has positive edge sign product, *consistent* if every circle has positive vertex sign product, and *harmonious* if the product of edge and vertex signs together is positive on every circle. When there are both edge and vertex signs, these three properties are all different, though any two of them imply the third. Balance of a signed graph, and consistency of a vertex-signed graph, are special cases of harmony if one assumes the missing vertex or edge signature is all positive.

A *Harary bipartition* of a balanced signed graph Σ is a bipartition $V = V_+ \cup V_-$ such that $V_+ \cup V_- = V$, $V_+ \cap V_- = \varnothing$, and an edge e is positive if and only if its endpoints belong to the same part V_s, $s \in \{+, -\}$. V_+ or V_- may be empty. This signed bipartition leads to a vertex signing μ such that $\mu(v) = +$ if $v \in V_+$ and $-$ if $v \in V_-$. By definition of the Harary bipartition, $\sigma(e_{vw}) = \mu(v)\mu(w)$ for every edge; it follows that $\sigma(P) = \mu(v)\mu(w)$ for every trail P from v to w. (Also, it does not matter which vertex set V_s is called the positive one.)

The *line graph of a signed graph* Σ, written $L = L(\Sigma)$, is defined as the vertex-signed graph $(L(G), \sigma)$ whose underlying graph is $L(G)$ and whose vertices are marked by the sign function σ of Σ. (There are several existing notions of line graph of a signed graph in which edges are signed instead of vertices; they are not related to this work.)

Definition 1. A signed graph Σ is *line consistent* if $L(\Sigma)$ is consistent. (This is the same as "S-consistency" in [2], where Σ is called S.)

Definition 2. A signed graph Σ is an *AAS graph* if it satisfies the following conditions (the *AAS conditions*) from [2, Theorem 2.1]:

- it is balanced;
- every supervalent vertex is totally positive;
- each trivalent vertex is totally positive, or it has two negative edges which belong to all circles through the vertex.

Theorem AAS ([2, Theorem 2.1])**.** *Σ is line consistent if and only if it is an AAS graph.*

The present work is based on this result. The necessity of the AAS conditions is easy. Thus, the purpose of our construction is to prove their sufficiency. The construction was inspired by the following observation (which is proved via Lemma 2):

Corollary 1 (of Theorem AAS)**.** *Let Σ be 2-connected of order at least 3. It is line consistent if and only if it is balanced and every negative edge has divalent endpoints.*

Since we know how to build 2-connected graphs by the process called ear adjunction (to be defined in Section 3, Construction AAS, Stage 1) and how to assemble them into arbitrary graphs, Corollary 1 suggests applying that knowledge to build line-consistent signed graphs. Thus, in Section 3 I show a procedure for constructing all AAS graphs; then in Section 4 I prove that every such graph is line consistent.

3. Construction AAS

The construction of AAS signed graphs has four stages, each of which may have several steps. The first two stages are separate basic constructions. The latter two assemble parts into a whole graph. I call the four stages **Construction AAS**.

We need some unusual terminology. Define a *big block* of G to be a maximal 2-connected subgraph that is not an isthmus; in other words, it is a maximal subgraph that contains a circle and has no cutpoints. A graph is a big block if it is 2-connected and has more than one edge. The set of isthmi, with their endpoints, is a forest; its components are the *tree blocks* of G. Thus, a *tree block* is a maximal connected subgraph with at least one edge, whose edges are isthmi; and it is not a block in the ordinary sense unless it has only one edge. A *major block* of a graph is either a big block or a tree block. A cutpoint is a *major cutpoint* if it is incident to a big block; otherwise it is a *minor cutpoint*. A minor cutpoint is incident only to edges of one tree block; a major cutpoint is incident to edges of more than one major block, at most one of which can be a tree block.

Stage 1. We construct AAS signed big blocks. An *ear* of G is an induced path whose endpoints are at least trivalent. Every big block other than a circle has an ear. It follows that any big block is constructible by successively adjoining ears, starting with a circle. (See [9, Theorem III.11] or [4, Proposition 3.1.3].) Here is the procedure we follow to build AAS big blocks:

(1) Take a circle C_0 and sign it any balanced way. This induces a Harary bipartition of the vertex set.
(2) Add an ear P so the endpoints p_0, p_l are totally positive. Sign the end edges of the ear positive and the internal edges so the resulting graph is balanced; that is, in any way such that $\sigma(P) = \mu(p_0)\mu(p_l)$. Extend the Harary bipartition to the new vertices.
(3) Continue till done.

Stage 2. Here is a four-step construction for AAS trees.

(1) Start with a tree T. Label vertices with their valencies.
(2) Delete all supervalent vertices, leaving T'.
(3) In T' choose a (possibly void) family of vertex-disjoint paths whose endpoints are at most divalent in T.
(4) In T, sign the path edges negative and the remaining edges positive.

Stage 3. Here is a construction for connected, separable AAS signed graphs. There are three steps. The only obligatory part is step (1).

(1) Take any set of AAS signed big blocks $\Sigma_1, \Sigma_2, \dots, \Sigma_l$ (where $l \geq 0$) and AAS signed trees $T_1, T_2, \dots, T_k$ (where $k \geq 0$), all on disjoint vertex sets. The disjoint union of the Σ_j and T_i is an AAS signed graph.
(2) Successively identify pairs of totally positive vertices in different components, such that at most one of the two vertices belongs to a tree T_i, as many times as desired. The result is clearly an AAS signed graph.
(3) Identify a divalent, totally negative vertex of some Σ_j (which may be contained in a larger component Υ_j formed in (2) or in previous applications of (3)) with a monovalent, totally positive vertex which belongs to some T_i that is not contained in Υ_j. (The valencies are in the graph as modified up to now; thus, neither of the two vertices can have been identified with any other vertex at a previous step.) Repeat as often as desired. Evidently, the result after each identification will be an AAS signed graph.

Stage 4. The (trivial) construction of a general AAS signed graph:

(1) Each component is constructed via Stages 1–3, or it is K_1.

Lemma 2. *In a 2-connected AAS signed graph, every trivalent vertex is totally positive.*

Proof. If two edges at a trivalent vertex are negative, by the AAS conditions the remaining edge cannot lie in a common circle with either of them. But in a 2-connected graph, any two edges are contained in some circle. □

Lemma 3. *Every AAS signed graph can be obtained by Construction AAS.*

Proof. Assume Σ is an AAS graph, hence balanced.

First, consider Σ that is 2-connected and has at least three vertices. If it is a circle, it is balanced and therefore is constructed in Stage 1. Otherwise, by Lemma 2 an ear P has totally positive endpoints. Thus, the rule of Stage 1 does generate Σ by adjoining P to $\Sigma \setminus P$.

Next, consider Σ that is a tree. By the AAS conditions, the negative subgraph is the disjoint union of paths in the underlying graph G, each of whose endpoints has degree at most 2 in G and none of whose internal vertices is supervalent. This is the kind of signed tree constructed in Stage 2.

Now consider a connected AAS graph Σ with more than one vertex. Each tree block T_i is AAS so it is constructible in Stage 2. Each big block of Σ is AAS, so it is constructible by Stage 1. We need to show that Σ is constructible from these units by Stage 3.

Let us examine a major cutpoint v. It must be at least trivalent. (If not, it would have only two edges, which are not in a block together; thus, each edge would be an isthmus and v would be a minor cutpoint.) It may belong to any positive number of big blocks Σ_j but at most one tree block T_i. Let Σ_1 be a big block incident with v. Replace v by two vertices, v_1 and v_2, and attach the edges of Σ_1 to v_1 and the other edges at v to v_2 (this is called *cleaving* v). This gives a new signed graph Υ which has one more component than Σ. The component Υ_1 that contains v_1, and $\Upsilon_2 = \Upsilon \setminus \Upsilon_1$, are AAS graphs.

If v is totally positive, and if Υ_1 and Υ_2 are produced by Construction AAS, then Σ is produced from Υ by amalgamating v_1 and v_2 at Stage 3(2).

If v is not totally positive, by the AAS conditions it must be trivalent and have two negative edges, e_1 and e_2, which are in a block together. Thus, e_1 and e_2 are in Σ_1, of which v is a divalent vertex, and the remaining edge f is an isthmus belonging to a tree block T_i. In the cleaved graph Υ, v_1 is divalent and totally negative and belongs to the big block Σ_1; and v_2 is a monovalent vertex which belongs to a tree block T_i. Thus, Σ can be constructed from Υ in Stage 3(3) by amalgamating v_1 and v_2.

Finally, it is plain that a disconnected AAS graph is constructed from its connected components at Stage 4. □

4. Proof of Theorem AAS

We show that Construction AAS always gives a line-consistent graph. For that we need a simple description of paths and circles in line graphs. There are two fundamental types. A *line path* or *line circle* in $L(G)$ is the line graph of a path or circle P in G; it is obtained by taking only the edges of P in the same order as in P. In a line circle, one repeats the final edge of P to get a circle in the line graph. A *vertex clique* in $L(G)$ consists of the edges incident with a single vertex of G, and a *vertex path* in $L(G)$ is a sequence of distinct members of a vertex clique. A *vertex circle* is the line graph of a circle in G. A *nontrivial* vertex path consists of at least three G-edges.

Concatenating two paths or circles P and P', expressed in vertex form as $P = \cdots z$ and $P' = a \cdots$, means producing a path PP' where either $z = a$ and P' simply continues from where P left off, or z and a are adjacent and PP' contains the connecting edge za. (This definition is slightly different from the usual one, in which one assumes $z = a$, but it is suitable to concatenation of line paths in $L(G)$.)

Lemma 4. *If C, a circle in $L(G)$, contains a nontrivial vertex path, the internal vertices of that path can be cut out, leaving a shorter circle.*

Proof. Let $C = g_1 g_2 \cdots g_k e \cdots fe$ where $g_1 g_2 \cdots g_k$ is a nontrivial vertex path. Since g_1 and g_k share a vertex, $C' = g_1 g_k e \cdots fe$ is a circle in $L(G)$. □

Lemma 5. *If Q, a path in $L(G)$, contains no nontrivial vertex path, then it has the form $L(R) = e_1 \cdots e_l$ for some trail $R = v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$ in G.*

Proof. Let $Q = e_1 e_2 \cdots e_l$ and let v_i be the common vertex of e_i and e_{i+1}. No v_i can equal v_{i+1} because that would form a nontrivial vertex path $e_i e_{i+1} e_{i+2}$. Therefore, defining v_0 and v_l by letting $e_1 = v_0 v_1$ and $e_l = v_{l-1} v_l$, the sequence $v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$ is a trail in G. □

Lemma 6. *The AAS conditions imply line consistency.*

Proof. We track the effect of each stage of the construction process in order to deduce line consistency. Write $L' = L(\Sigma')$.

Stage 1. First, we track the process of ear adjunction.

(1) $L(C_0)$ is consistent by construction.

(2) Call the signed graph without the ear Σ and with the ear call it Σ'. Assume Σ is line consistent. The ear is $P = uf \cdots gv$ where $u, v \in V$ and $f, g \in E'$. It has length at least 1, which means that f and g may be the same edge. Let a, a' and b, b', respectively, be the edges of Σ incident with u and v. By construction Σ' is balanced. Consider a circle C' in L' that does not lie in L.

Suppose C' contains a nontrivial vertex path $g_1 g_2 \cdots g_{k-1} g_k$. By Lemma 2 $g_2, \ldots, g_{k-1}$ are positive, so they can be cut out to give a shorter circle with the same sign. Thus, we may assume C' contains no nontrivial vertex path. Since a vertex circle is all positive, we may also assume C' is not a vertex circle.

If C' contained edge f without containing a path of the form afb where a is incident with u and b with v, then it would contain a nontrivial vertex path of the form afa' where a, a' are incident with u. Thus, C' contains $L(P) = f \cdots g$ as a subpath, so $C' = aL(P)bQa$ where bQa is a path in L of the form $L(R)$ by Lemma 5. This is positive because $\sigma(L(P)) = \sigma(P) = \mu(u)\mu(v)$ by construction and $\sigma(bQa) = \sigma(R) = \mu(u)\mu(v)$ by the nature of a Harary bipartition.

Thus, C' is positive in every case, so Σ' is line consistent.

By induction, the signed graph resulting from Stage 1 is line consistent.

Stage 2. A circle in $L(T)$ can only consist of edges incident to a vertex of T. The AAS conditions force such a circle to be consistent. Thus, L is consistent if Σ is a signed tree.

Stage 3. In Stage 3 we examine the effect of one vertex identification. There are two cases, corresponding to Steps (2) and (3). Let Σ be the result of identifying $v_1 \in V(\Upsilon_1)$ and

$v_2 \in V(\Upsilon_2)$ to $v \in V$ (where the Υ_i are components of the graph before this identification), and let C be a circle in L.

(1) The big blocks and tree blocks are line consistent by previous stages of the proof.

(2) Suppose v_1 and v_2 are totally positive. Assume neither v_i is monovalent; the monovalent case is treated below in (3). Let N_i be the set of edges of Υ_i incident with v and let $N = N_1 \cup N_2$. A circle C that is not contained in L_1 must contain edges of both N_1 and N_2; but we may assume it contains no nontrivial path in the vertex clique of v since v is totally positive.

Then C must be a concatenation $C = QQ' \cdots Q^{(k)} f$, where each $Q^{(j)}$ is a path in $L(\Upsilon_1)$ or $L(\Upsilon_2)$ whose endpoints, but none of its internal vertices, are edges in N. Let $f^{(j)}$ be the first edge of $F^{(j)}$.

Now, $\sigma(C) = \sigma(Q)\sigma(Q') \cdots \sigma(Q^{(k)}$. Each $Q^{(j)} f^{(j)}$ is a circle in $L(\Upsilon_1)$ or $L(\Upsilon_2)$, hence consistent, by induction. Since all $\sigma(Q^{(j)}) = +$ C is consistent.

(3) Suppose v_2 is monovalent, so its only incident edge is an isthmus f. The only way C can contain an isthmus is as an internal vertex of a nontrivial vertex path, or if C is a vertex circle. Apply this observation to f. If C is a vertex circle, it contains at least three edges incident to v, all of which are positive in Step (2), and two of which are negative in Step (3); thus, C is positive. If C is not a vertex circle, since f is positive it can be cut out of C to form a shorter circle C' of the same sign. Then C' is a circle in $L(\Upsilon_1)$ and is therefore positive by induction.

We conclude that L is consistent.

Stage 4 is trivial.

The analysis of these stages completes the proof. □

5. Future developments

What is missing is a direct, non-recursive description of line-consistent signed graphs. That is where I expect Tutte's 3-decomposition to come in, because consistency of a 3-connected signed graph is extremely simple.

Corollary 7 (of Theorem AAS)**.** *Let Σ be 3-connected of order at least 4, or just 2-connected without divalent vertices. Then Σ is line consistent if and only if it is all positive.*

It would add considerably to the value and interest of all the results mentioned in this article if they could be extended to the situation in which $L(G)$ has not only the vertex signature σ but also an arbitrary edge signature ξ. The question then is to characterize the choices of σ and ξ that make the line graph $L(G)$ with its vertex and edge signs harmonious. As Hoede's theorem applies to graphs with both vertex and edge signs, the proof may, perhaps, be achieved along the lines of the original proof of Theorem AAS. Or, since a signed graph is determined up to switching by the list of balanced induced circles, Truemper's characterization of the possible patterns of balanced induced circles [8] might be the tool necessary to solve the problem of line harmony.

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