Dowling Geometries of Multiary Quasigroups

Thomas Zaslavsky
Binghamton University (SUNY)

Sources:

Erratum. *ibid.* **15** (1973), 211.

Dowling lattice/geometry/matroid (of rank $r$) of a group $\mathcal{G}$:

$$Q_r(\mathcal{G})$$

Dowling’s abstract definition (lattice):

Poset of homogeneously $\mathcal{G}$-labelled partial partitions of an $r$-set $V$.

Geometrical definition (matroid):

Set of homogeneous vectors over $\mathcal{G}$ with 1 or 2 non-zero coordinates.

Graphical definition (matroid):

Take the complete graph $K_r$.
Replace each edge by a ‘pencil’ of parallel edges labelled invertibly by $\mathcal{G}$ (gains).
Add a loop at each vertex labelled by a non-identity element (gain).
Take the frame matroid $G(\mathcal{G}K_r^*)$.
Frame circuits:
- Circles with identity gain (balanced circles).
- Theta graphs with no balanced circle.
- Barbells and figure-eights with no balanced circle.

Advantage: minors by graph theory.
Beautiful properties

Universality, similarly to projective spaces over a skew field (Kahn–Kung).

Contractions are Dowling matroids (upper intervals are Dowling lattices).

Superbly easy computation of invariants from gain-graph coloring:

\[
p_{Q_r(G)}(y) = |G|^r \cdot \left( \frac{y - 1}{|G|} \right)_r = |G|^r \cdot \chi_{K_r} \left( \frac{y - 1}{|G|} \right)
\]

Other graph expansions:

\[
p_{G \Gamma^*}(y) = |G|^r \cdot \chi_{\Gamma} \left( \frac{y - 1}{|G|} \right)
\]

Thus, \(G(\mathcal{G} \Gamma^*)\) is a graph-based analog of Dowling matroid.
Quasigroup Dowling matroid

Quasigroup: a group without associativity:  \( \mathfrak{Q} = (\mathfrak{Q}, \cdot) \) such that
\[
xy = z \quad \text{has unique solvability.}
\]

Analogously, rewrite a group operation as \( x \cdot y = z \).
Reinterpret \( Q_3(\mathfrak{S}) \): A balanced circle has \( xy = z \).
Apply to \( \mathfrak{Q} \): you get balanced circles, a frame matroid, and the Dowling plane of \( \mathfrak{Q} \):
\[
Q_3(\mathfrak{Q}) = G(\mathfrak{Q} K_3^*)
\]

*This has no extension to higher rank!*

**Question:** Are there higher-rank generalized Dowling matroids?

Graphic version: Maximal biased expansion of a graph.
- Replace each edge in \( \Gamma \) by a ‘pencil’ of parallel edges.
- Define a class of balanced circles.
- Add an unbalanced loop at each vertex.
- Take the frame matroid \( G(m \cdot \Gamma^*) \).

**Question:** Are there maximal examples other than Dowling matroids of groups and quasigroups?
Multiary quasigroups

An $n$-ary quasigroup $Q$: an $n$-ary operation such that

$$(x_1 x_2 \cdots x_n) = x_0$$

has unique solvability.

Example: Iterated group operation:

$$(x_1 x_2 \cdots x_n) = x_0 \text{ if } x_1 \cdot x_2 \cdots \cdot x_n.$$  

Factors in all possible ways:

$$(x_1 x_2 \cdots x_n) = (x_1 \cdots x_i [x_{i+1} \cdots x_j] x_{j+1} \cdots x_n)$$

**Theorem 1** (essentially: Aczél, Belousov, and Hosszú; Kahn and Kung).

*If $Q$ factors in all possible ways, it is an iterated group operation.*

Example: $(xy)z = (xyz) = x(yz) \implies$ group.

The opposite extreme: $Q$ is *irreducible* if it has no factorizations.
Partial factorization and the factorization graph

Draw $C_{n+1} = v_0 e_1 v_1 e_2 \cdots e_n v_n e_0 v_0$. In $C_{n+1}$:

$$v_i \leftrightarrow x_i.$$ 

Edge $e_{ij} \leftrightarrow$ factorization $(x_1 \cdots x_i [x_{i+1} \cdots x_j] x_{j+1} \cdots x_n)$.

This is the factorization graph $\Phi(\Omega)$.

**Theorem 2** (Zaslavsky).
$\Omega$ is an iterated group if and only if $\Phi(\Omega)$ is 3-connected.

Proof: Intransitive combinatorial homotopy within the biased expansion graph.

**Theorem 3** (Zaslavsky).
Every $n$-ary quasigroup with $n \geq 3$ is (in a unique way) the composition of irreducible (quasigroups and) multiary quasigroups and iterated groups.

Proof: Tutte’s 3-decomposition theorem.
**Generalized Dowling matroids**

A *generalized Dowling geometry* is a frame matroid

\[ Q_r = G((\text{biased expansion of } \Gamma)^*) \]

which cannot be extended to another such frame matroid, on a supergraph of \( \Gamma \), on the same vertex set.

Example: \( \Gamma = K_r \), biased expansion = \( \mathfrak{G}K_r \) of a group.

Example: \( \Gamma = K_3 \), biased expansion = \( \mathfrak{Q}K_3 \) of a quasigroup.

Example: \( \Gamma = C_r \), biased expansion = \( \mathfrak{Q}C_r \) of an irreducible \( r + 1 \)-ary quasigroup.

**Theorem 4** (Corollary of Theorem 3).

*Universal example:*

(a) \( \Omega = 2\text{-amalgamation of complete graphs and circles, the complete graphs expanded by groups and the circles by non-group multiary quasigroups.} \)

(b) \( Q_r = G(\Omega^*) \).