The Dynamics of the Forest Graph Operator

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Abstract

In 1966, Cummins introduced the “tree graph”: the tree graph $T(G)$ of a graph $G$ (possibly infinite) has all its spanning trees as vertices, and distinct such trees correspond to adjacent vertices if they differ in just one edge. i.e., two spanning trees $T_1$ and $T_2$ are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The tree graph of a connected graph need not be connected. To obviate this we define the “forest graph”: let $G$ be a labeled graph of order $\alpha$, finite or infinite, and let $\mathcal{N}(G)$ be the set of all labeled maximal forests of $G$. The forest graph of $G$, denoted by $\mathcal{F}(G)$, is the graph with vertex set $\mathcal{N}(G)$ in which two maximal forests $F_1, F_2$ of $G$ form an edge if and only if they differ exactly by one edge, i.e., $F_2 = F_1 - e + f$ for some edges $e \in F_1$ and $f \notin F_1$.

Using the theory of cardinal numbers, Zorn’s lemma, transfinite induction, the axiom of choice and the well-ordering principle, we determine the $\mathcal{F}$-convergence, $\mathcal{F}$-divergence, $\mathcal{F}$-depth and $\mathcal{F}$-stability of any graph $G$. In particular it is shown that a graph $G$ (finite or infinite) is $\mathcal{F}$-convergent if and only if $G$ has at most one cycle of length 3. The $\mathcal{F}$-stable graphs are precisely $K_3$ and $K_1$. The $\mathcal{F}$-depth of any graph $G$ different from $K_3$ and $K_1$ is finite.

\textbf{Keywords:} Graph operator, Forest graph, $\mathcal{F}$-convergence, $\mathcal{F}$-divergence, $\mathcal{F}$-depth, $\mathcal{F}$-stable.
1 Introduction

Let $G$ be a labeled graph of order $\alpha$, finite or infinite. (All our graphs are labeled.) A spanning tree of $G$ is a connected, acyclic, spanning subgraph of $G$; it exists if and only if $G$ is connected. Any acyclic subgraph of $G$, connected or not, is called a forest of $G$. A forest $F$ of $G$ is said to be maximal if there is no forest $F'$ of $G$ such that $F$ is a proper subgraph of $F'$. The tree graph $T(G)$ of $G$ has all the spanning trees of $G$ as vertices; distinct such trees are adjacent vertices if they differ in just one edge [10, 13]; i.e., two spanning trees $T_1$ and $T_2$ are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The iterated tree graphs of $G$ are defined by $T_0(G) = G$ and $T^n(G) = T(T^{n-1}(G))$ for $n > 0$. There are several results on tree graphs. See, [1, 14, 9] for connectivity of the tree graph, [7, 11, 4, 6, 3] for bounds on the order of $T(G)$ (that is, on the number of spanning trees of $G$), [2, 12] for hamilton circuits in a tree graph.

There is one difficulty with iterating the tree graph operator. The tree graph of an infinite connected graph need not be connected [2, 12], so $T^2(G)$ may be undefined. For example, $T(K_{\aleph_0})$ is disconnected (see Corollary 2.5 in this paper; $\aleph_0$ denotes the cardinality of the set $\mathbb{N}$ of natural numbers); therefore $T^2(K_{\aleph_0})$ is not defined. To obviate this difficulty with iterated tree graphs, and inspired by the tree graph operator $T$, we define a forest graph operator. Let $N(G)$ be the set of all maximal forests of $G$. The forest graph of $G$, denoted by $F(G)$, is the graph with vertex set $N(G)$ in which two maximal forests $F_1$, $F_2$ form an edge if and only if they differ by exactly one edge. The forest graph operator on graphs, $G \mapsto F(G)$, is denoted by $F$. Zorn’s lemma implies that every connected graph contains a spanning tree (see [5]); similarly, every graph has a maximal forest. Hence, the forest graph always exists. Since when $G$ is connected, maximal forests are the same as spanning trees, then $F(G) = T(G)$; that is, the tree graph is a special case of the forest graph. We write $F^2(G)$ to denote $F(F(G))$, and in general $F^n(G) = F(F^{n-1}(G))$ for $n \geq 1$, with $F^0(G) = G$.

Definition 1.1. A graph $G$ is said to be $F$-convergent if $\{F^n(G) : n \in \mathbb{N}\}$ is finite; otherwise $G$ is $F$-divergent.

Definition 1.2. A graph $H$ is said to be an $F$-root of a graph $G$ if $F(H)$ is isomorphic to $G$, $F(H) \cong G$. The $F$-depth of $G$ is

$$\text{sup}\{n \in \mathbb{N} : G \cong F^n(H) \text{ for some graph } H\}.$$ 

The $F$-depth of a graph $G$ that has no $F$-root is said to be zero.
Definition 1.3. A graph $G$ is said to be $\mathbf{F}$-periodic if there exists a positive integer $n$ such that $\mathbf{F}^n(G) = G$. The least such integer is called the $\mathbf{F}$-periodicity of $G$. If $n = 1$, $G$ is called $\mathbf{F}$-stable.

This paper is organized as follows. In Section 2 we give some basic results. In later sections, using Zorn’s lemma, transfinite induction, the well ordering principle and the theory of cardinal numbers, we study the number of $\mathbf{F}$-roots and determine the $\mathbf{F}$-convergence, $\mathbf{F}$-divergence, $\mathbf{F}$-depth and $\mathbf{F}$-stability of any graph $G$. In particular we show that: i) A graph $G$ is $\mathbf{F}$-convergent if and only if $G$ has at most one cycle of length 3. ii) The $\mathbf{F}$-depth of any graph $G$ different from $K_3$ and $K_1$ is finite. iii) The $\mathbf{F}$-stable graphs are precisely $K_3$ and $K_1$. iv) A graph that has one $\mathbf{F}$-root has innumerably many, but only some $\mathbf{F}$-roots are important.

2 Preliminaries

For standard notation and terminology in graph theory we follow Diestel [5] and Prisner [10].

Some elementary properties of infinite cardinal numbers that we use are (see, e.g., Kamke [8]):

1. $\alpha + \beta = \alpha \cdot \beta = \max(\alpha, \beta)$ if $\alpha, \beta$ are cardinal numbers and $\beta$ is infinite. In particular, $2 \cdot \beta = \aleph_0 \cdot \beta = \beta$.

2. $\beta^n = \beta$ if $\beta$ is an infinite cardinal and $n$ is a positive integer.

3. $\beta < 2^\beta$ for every cardinal number.

4. The number of finite subsets of an infinite set of cardinality $\beta$ is equal to $\beta$.

We consider finite and infinite labeled graphs without multiple edges or loops. An isthmus of a graph $G$ is an edge $e$ such that deleting $e$ divides one component of $G$ into two of $G - e$. Equivalently, an isthmus is an edge that belongs to no cycle. Each isthmus is in every maximal forest, but no non-isthmus is.

Let $\mathcal{C}(G)$, $\mathcal{C}'(G)$ and $\mathcal{N}(G)$ denote the set of all possible cycles, the set of all maximal sets of pairwise edge disjoint cycles and the set of all maximal forests of a graph $G$, respectively. Note that a maximal forest of $G$ consists of a spanning tree in each component of $G$. A fundamental fact, whose proof is similar to that of the existence of a maximal forest, is the following forest extension lemma:
Lemma 2.1. In any graph $G$, every forest is contained in a maximal forest.

The existence of a maximal set of pairwise edge disjoint cycles is justified in a similar manner.

Lemma 2.2. If $G$ is a complete graph of infinite order $\alpha$, then $|\mathcal{F}(G)| = 2^\alpha$.

Proof: Let $G = (V, E)$ be a complete graph of order $\alpha$ ($\alpha$ infinite), i.e., $G = K_\alpha$. Let $v_1, v_2$ be two vertices of $G$ and $V' = V \setminus \{v_1, v_2\}$. Then for every $A \subseteq V'$ there is a spanning tree $T_A$ such that every vertex of $A$ is adjacent only to $v_1$ and every vertex of $V' \setminus A$ is adjacent only to $v_2$. It is easy to see that $T_A \neq T_B$ whenever $A \neq B$. As the cardinality of the power set of $V'$ is $2^\alpha$, there are at least $2^\alpha$ spanning trees of $G$. Since $G$ is connected, the maximal forests are the spanning trees; therefore $|\mathcal{F}(G)| \geq 2^\alpha$. Since the degree of each vertex is $\alpha$ and $G$ contains $\alpha$ vertices, the total number of edges in $G$ is $\alpha \cdot \alpha = \alpha$. The edge set of a maximal forest of $G$ is a subset of $E$ and the number of all possible subsets of $E$ is $2^\alpha$. Therefore, $G$ has at most $2^\alpha$ maximal forests, i.e., $|\mathcal{F}(G)| \leq 2^\alpha$. Hence $|\mathcal{F}(G)| = 2^\alpha$. $\blacksquare$

For two maximal forests of $G$, $F_1$ and $F_2$, let $d(F_1, F_2)$ denote the distance between them in $\mathcal{F}(G)$. We connect this distance to the number of edges by which $F_1, F_2$ differ; the result is elementary but we could not find it anywhere in the literature. We say $F_1, F_2$ differ by $l$ edges if $|E(F_1) \setminus E(F_2)| = |E(F_2) \setminus E(F_1)| = l$.

Lemma 2.3. Let $l$ be a natural number. For two maximal forests $F_1, F_2$ of a graph $G$, if $|E(F_1) \setminus E(F_2)| = l$, then $|E(F_2) \setminus E(F_1)| = l$. Furthermore, $F_1, F_2$ differ by exactly $l$ edges if and only if $d(F_1, F_2) = l$.

We cannot apply to an infinite graph the simple proof for finite graphs, in which the number of edges in a maximal forest is given by a formula. Therefore, we prove the lemma by edge exchange.

Proof: We prove the first part by induction on $l$. Let $F_1, F_2$ be maximal forest of $G$ and let $E(F_1) \setminus E(F_2) = \{e_1, e_2, \ldots, e_k\}$, $E(F_2) \setminus E(F_1) = \{e_1, e_2, \ldots, e_l\}$. If $l = 0$ then $k = 0 = l$ because $F_2 = F_1$. Suppose $l > 0$; then $k > 0$ also. Deleting $e_l$ from $F_2$ divides a tree of $F_2$ into two trees. Since these trees are in the same component of $G$, there is an edge of $F_1$ that connects them; this edge is not $e_1$ so it is not in $F_2$; therefore, it is an $e_i$, say $e_i'$. Let $F_2' = F_2 - e_l + e_i'$. Then $E(F_1) \setminus E(F_2') = \{e_1', e_2', \ldots, e_{k-1}'\}$, $E(F_2') \setminus E(F_1) = \{e_1, e_2, \ldots, e_{l-1}\}$. By induction, $k - 1 = l - 1$.

We also prove the second part by induction on $l$. Assume $F_1, F_2$ differ by exactly $l$ edges and define $F_2'$ as above. If $l = 0, 1$, clearly $d(F_1, F_2) = l$. Suppose $l > 1$. In
a shortest path from $F_1$ to $F_2$, whose length is $d(F_1, F_2)$, each successive edge of the path can increase the number of edges not in $F_1$ by at most 1. Therefore, $F_1$ and $F_2$ differ by at most $d(F_1, F_2)$ edges. That is, $l \leq d(F_1, F_2)$. Conversely, $d(F_1, F'_2) = l - 1$ by induction and there is a path in $F(G)$ from $F_1$ to $F'_2$ of length $l - 1$, then continuing to $F_2$ and having total length $l$. Thus, $d(F_1, F_2) \leq l$.

From the above lemma we have two corollaries.

**Corollary 2.4.** For any graph $G$, $F(G)$ is connected if and only if any two maximal forests of $G$ differ by at most a finite number of edges.

**Corollary 2.5.** If $G = K_\alpha$, $\alpha$ infinite, then $F(G)$ is disconnected.

**Lemma 2.6.** Let $G$ be a graph with $\alpha$ vertices and $\beta$ edges and with no isolated vertices. If either $\alpha$ or $\beta$ is infinite, then $\alpha = \beta$.

**Proof:** We know that $|E(G)| \leq |V(G)|^2$, i.e., $\beta \leq \alpha^2$ so if $\beta$ is infinite, $\alpha$ must also be infinite. We also know, since each edge has two endpoints, that $|V(G)| \leq 2|E(G)|$, i.e., $\alpha \leq 2\beta$ so if $\alpha$ is infinite, then $\beta$ must be infinite. Now assuming both are infinite, $\alpha^2 = \alpha$ and $2\beta = \beta$, hence $\alpha = \beta$.

The following lemmas are needed in connection with $F$-convergence and $F$-divergence in Section 5 and $F$-depth in Section 6.

**Lemma 2.7.** Let $G$ be a graph. If $K_n$ (for finite $n \geq 2$) is a subgraph of $G$, then $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $F(G)$.

**Proof:** Let $G$ be a graph such that $K_n$ (for $n \geq 2$, finite) is a subgraph of $G$ with vertex labels $v_1, v_2, \ldots, v_n$. Then there is a path $L = v_1, v_2, \ldots, v_n$ of order $n$ in $G$. Let $F$ be a maximal forest of $G$ such that $F$ contains the path $L$. In $F$ if we replace the edge $v_{[n/2]}v_{[n/2]+1}$ by any other edge $v_iv_j$ where $i = 1, \ldots, [n/2]$ and $j = [n/2] + 1 \ldots n$, we get a maximal forest $F_{ij}$. Since there are $\lfloor n^2/4 \rfloor$ such edges $v_iv_j$, there are $\lfloor n^2/4 \rfloor$ maximal forests $F_{ij}$ (of which one is $F$). Any two forests $F_{ij}$ differ by one edge. It follows that they form a complete subgraph in $F(G)$. Therefore $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $F(G)$.

**Lemma 2.8.** If $G$ has a cycle of (finite) length $n$ with $n \geq 3$, then $F(G)$ contains $K_n$.

**Proof:** Suppose that $G$ has a cycle $C_n$ of length $n$ with edge set $\{e_1, e_2, \ldots, e_n\}$. Let $P_i = C_n - e_i$ for $i = 1, 2, \ldots, n$ and let $F_1$ be a maximal forest of $G$ containing the
path $P_i$. Define $F_i = F_i \setminus P_i \cup P_i$ for $i = 2, 3, \ldots, n$. These $F_i$'s are maximal forests of $G$ and any two of them differ by exactly one edge, so they form a complete graph $K_n$ in $F(G)$.

Lemma 2.9. Suppose that $G$ contains $K_n$, where $n \geq 3$. Then $F^2(G)$ contains $K_{n^2-2}$.

Proof: Cayley’s formula states that $K_n$ has $n^{n-2}$ spanning trees. Cummins [2] proved that the tree graph of a finite connected graph is Hamiltonian. Therefore, $F(K_n)$ contains $C_{n^2-2}$. Let $F_{T_0}$ be a spanning tree of $G$ that extends one of the spanning trees $T_0$ of the $K_n$ subgraph. Replacing the edges of $T_0$ in $F_{T_0}$ by the edges of any other spanning tree $T$ of $K_n$, we have a spanning tree $F_T$ that contains $T$. The $F_T$'s for all spanning trees $T$ of $K_n$ are $n^{n-2}$ spanning trees of $G$ that differ only within $K_n$; thus, the graph of the $F_T$'s is the same as the graph of the $T$'s, which is Hamiltonian. That is, $F(G)$ contains $C_{n^2-2}$. By Lemma 2.8, $F^2(G)$ contains $K_{n^2-2}$.

Lemma 2.10. If $G$ has two edge disjoint triangles, then $F^2(G)$ contains $K_9$.

Proof: Suppose that $G$ has two edge disjoint triangles whose edges are $e_1, e_2, e_3$ and $f_1, f_2, f_3$, respectively. The union of the triangles has exactly 9 maximal forests $F'_{ij}$, obtained by deleting one $e_i$ and one $f_j$ from the triangles. Extend $F'_{11}$ to a maximal forest $F_{11}$ and let $F_{ij}$ be the maximal forest $F_{11} \setminus E(F'_{11}) \cup F_{ij}$, for each $i, j = 1, 2, 3$. The 9 maximal forests $F'_{ij}$, and consequently the maximal forests $F_{ij}$ in $F(G)$, form a Cartesian product graph $C_3 \times C_3$, which contains a cycle of length 9. By Lemma 2.8, $F^2(G)$ contains $K_9$.

We now show that repeated application of the forest graph operator to many graphs creates larger and larger complete subgraphs.

Lemma 2.11. If $G$ has a cycle of (finite) length $n$ with $n \geq 4$ or it has two edge disjoint triangles, then for any finite $m \geq 2$, $F^m(G)$ contains $K_{m^2}$.

Proof: We prove this lemma by induction on $m$.

Case 1: Suppose that $G$ has a cycle $C_n$ of length $n$ ($n \geq 4$, $n$ finite). By Lemma 2.8, $F(G)$ contains $K_n$ as a subgraph, which implies that $F(G)$ contains $K_4$. By Lemma 2.9, $F^3(G)$ contains $K_{16}$ and in particular it contains $K_{32}$.

Case 2: Suppose that $G$ has two edge disjoint triangles. By Lemma 2.10 $F^2(G)$ contains $K_9$ as a subgraph. It follows by Lemma 2.7 that $F^3(G)$ contains $K_{192/41} = K_{20}$ as a subgraph. This implies that $F^3(G)$ contains $K_{32}$ as a subgraph.
By Cases 1 and 2 it follows that the result is true for \( m = 1, 2, 3 \). Let us assume that the result is true for \( m = l \geq 3 \), i.e., that \( F^l(G) \) contains \( K_{l^2} \) as a subgraph. By Lemma 2.7 it follows that \( F(F^l(G)) \) has a subgraph \( K_{[l^2/4]} \). Since \( [l^2/4] > (l+1)^2 \), it follows that \( F^{l+1}(G) \) contains \( K_{(l+1)^2} \). By the induction hypothesis \( F^m(G) \) contains \( K_{m^2} \), for any finite \( m \geq 2 \).

With Lemma 2.9 it is clearly possible to prove a much stronger lower bound on complete subgraphs of iterated forest graphs, but Lemma 2.11 is good enough for our purposes.

**Lemma 2.12.** A forest graph that is not \( K_1 \) has no isolated vertices and no isthmi.

**Proof:** Let \( G = F(H) \) for some graph \( H \). Consider a vertex \( F \) of \( G \), that is, a maximal forest in \( H \). Let \( e \) be an edge of \( F \) that belongs to a cycle \( C \) in \( H \). Then there is an edge \( f \) in \( C \) that is not in \( F \) and \( F' = F - e + f \) is a second maximal forest that is adjacent to \( F \) in \( G \). Since \( C \) has length at least 3, it has a third edge \( g \). If \( g \) is not in \( F \), let \( F'' = F - e + g \). If \( g \) is in \( F \), let \( F'' = F - g + f \). In both cases \( F'' \) is a maximal forest that is adjacent to \( F \) and \( F' \). Thus, \( F \) is not isolated and the edge \( FF'' \) in \( G \) is not an isthmus.

Suppose \( F, F' \in \mathfrak{N}(H) \) are adjacent in \( G \). That means there are edges \( e \in E(F) \) and \( e' \in E(F') \) such that \( F' = F - e + e' \). Thus, \( e \) belongs to the unique cycle in \( F + e' \). As shown above, there is an \( F'' \in \mathfrak{N}(H) \) that forms a cycle with \( F \) and \( F' \). Therefore the edge \( FF'' \) of \( G \) is not an isthmus.

Let \( F \in \mathfrak{N}(H) \) be an isolated vertex in \( G \). If \( H \) has an edge \( e \) not in \( F \), then \( F + e \) contains a cycle so \( F \) has a neighboring vertex in \( G \), as shown above. Therefore, no such \( e \) can exist; in other words, \( H = F \) and \( G \) is \( K_1 \).

**3 Basic Properties of an Infinite Forest Graph**

We now present a crucial foundation for the proof of the main theorem in Section 5. The *cyclomatic number* \( \beta_1(G) \) of a graph \( G \) can be defined as the cardinality \( |E(G) \setminus E(F)| \) where \( F \) is a maximal forest of \( G \).

**Proposition 3.1.** Let \( G \) be a graph such that \( |\mathfrak{C}(G)| = \beta \), an infinite cardinal number. Then:

i) \( \beta_1(G) = \beta \) and \( \beta_1(F(G)) = 2^{\beta} \).
ii) Both the order of $F(G)$ and its number of edges equal $2^\beta$. Both the order and the number of edges of $G$ equal $\beta$, provided that $G$ has no isolated vertices and no isthmi.

iii) $F(G)$ is $\beta$-regular.

iv) The order of any connected component of $F(G)$ is $\beta$.

v) $F(G)$ has exactly $2^\beta$ components.

vi) Every component of $F(G)$ has exactly $\beta$ cycles.

vii) $|\mathcal{C}(F(G))| = 2^\beta$.

**Proof:** Let $G$ be a graph with $|\mathcal{C}(G)| = \beta$ ($\beta$ infinite).

i) Let $F$ be a maximal forest of $G$. The number of cycles in $G$ is not more than the number of finite subsets of $E(G) \setminus E(F)$. This number is finite if $E(G) \setminus E(F)$ is finite, but it cannot be finite because $|\mathcal{C}(G)|$ is infinite. Therefore $E(G) \setminus E(F)$ is infinite and the number of its finite subsets equals $|E(G) \setminus E(F)| = \beta_1(G)$. Thus, $\beta_1(G) \geq |\mathcal{C}(G)|$. The number of cycles is at least as large as the number of edges not in $F$, because every such edge makes a different cycle with $F$. Thus, $|\mathcal{C}(G)| \geq \beta_1(G)$. It follows that $\beta_1(G) = |\mathcal{C}(G)| = \beta$. Note that this proves $\beta_1(G)$ does not depend on the choice of $F$.

The value of $\beta_1(F(G))$ follows from this and part (vii).

ii) For the first part, let $F$ be a maximal forest of $G$ and let $F_0$ be a maximal forest of $G \setminus E(F)$. As $G \setminus E(F)$ has $\beta_1(G) = \beta$ edges by part (i), it has $\beta$ non-isolated vertices by Lemma 2.6. $F_0$ has the same non-isolated vertices, so it too has $\beta$ edges.

Any edge set $A \subseteq F_0$ extends to a maximal forest $F_A$ in $F \cup A$. Since $F_A \setminus F = A$, the $F_A$’s are distinct. Therefore, there are at least $2^\beta$ maximal forests in $F_0 \cup F$. The maximal forest $F$ consists of a spanning tree in each component of $G$; therefore, the vertex sets of components of $F$ are the same as those of $G$, and so are those of $F_0 \cup F$. Therefore, a maximal forest in $F_0 \cup F$, which consists of a spanning tree in each component of $F_0 \cup F$, contains a spanning tree of each component of $G$.

We conclude that a maximal forest in $F_0 \cup F$ is a maximal forest of $G$ and hence that there are at least $2^\beta$ maximal forests in $G$, i.e., $|\mathcal{N}(G)| \geq 2^\beta$. Since $G$ is a subgraph of $K_\beta$, and since $|\mathcal{N}(K_\beta)| = 2^\beta$ by Lemma 2.2, we have $|\mathcal{N}(G)| \leq 2^\beta$. Therefore $|\mathcal{N}(G)| = 2^\beta$. That is, the order of $F(G)$ is $2^\beta$. By Lemmas 2.12 and 2.6, that is also the number of edges of $F(G)$.
For the second part, note that $G$ has infinite order or else $\beta_1(G)$ would be finite. If $G$ has no isolated vertices and no isthmi, then $|V(G)| = |E(G)|$ by Lemma 2.6. By part (i) there are $\beta$ edges of $G$ outside a maximal forest; hence $\beta \leq |E(G)|$.

Since every edge of $G$ is in a cycle, by the axiom of choice we can choose a cycle $C(e)$ containing $e$ for each edge $e$ of $G$. Let $\mathcal{C} = \{C(e) : e \in E(G)\}$. The total number of pairs $(f, C)$ such that $f \in C \in \mathcal{C}$ is no more than $\aleph_0, |\mathcal{C}| \leq \aleph_0, |\mathcal{C}(G)| = \aleph_0$. $\beta = \beta$. This number of pairs is not less than the number of edges, so $|E(G)| \leq \beta$. It follows that $G$ has exactly $\beta$ edges.

iii) Let $F$ be a maximal forest of $G$. By part (i), $|E(G) \setminus E(F)| = \beta$. By adding any edge $e$ from $E(G) \setminus E(F)$ to $F$ we get a cycle $C$. Removing any edge other than $e$ from the cycle $C$ gives a new maximal forest which differs by exactly one edge with $F$. The number of maximal forests we get in this way is $\beta_1(G)$ because there are $\beta_1(G)$ ways to choose $e$ and a finite number of edges of $C$ to choose to remove, and $\beta_1(G)$ is infinite. Thus we get $\beta$ maximal forests of $G$, each of which differs by exactly one edge with $F$. Every such maximal forest is generated by this construction. Therefore, the degree of any vertex in $F(G)$ is $\beta$.

iv) Let $A$ be a connected component of $F(G)$. As $F(G)$ is $\beta$-regular by part (iii), it follows that $|V(A)| \geq \beta$. Fix a vertex $v$ in $A$ and define the $n$th neighborhood $D_n = \{v' : d(v, v') = n\}$ for each $n \in \mathbb{N}$. Since every vertex has degree $\beta$, $|D_0| = 1$, $|D_1| = \beta$ and $|D_k| \leq \beta|D_{k-1}|$. Thus, by induction on $n$, $|D_n| \leq \beta$ for $n > 0$.

Since $A$ is connected, it follows that $V(A) = \bigcup_{n \in \mathbb{N} \setminus \{0\}} D_n$, i.e., $V(A)$ is the countable union of sets of order $\beta$. Therefore $|A| = \beta$, as $|\mathbb{N}|, \beta' = \beta'$. Hence any connected component of $F(G)$ has $\beta$ vertices.

v) By parts (ii, iv) the order of $F(G)$ is $2^\beta$ and the order of each component of $F(G)$ is $\beta$. Since $|F(G)| = 2^\beta$, $F(G)$ has at most $2^\beta$ components. Suppose that $F(G)$ has $\beta'$ components where $\beta' < 2^\beta$. As each component has $\beta$ vertices, it follows that $F(G)$ has order at most $\beta', \beta = \max\{\beta', \beta\}$. This is a contradiction to part (ii). Therefore $F(G)$ has exactly $2^\beta$ components.

vi) Let $A$ be a component of $F(G)$. Since it is infinite, by part (iv) and Lemma 2.6 it has exactly $\beta$ edges. Suppose that $|\mathcal{C}(A)| = \beta'$. Then $\beta'$ is at most the number of finite subsets of $E(A)$, which is $\beta$ since $|E(A)| = \beta$ is infinite; that is, $\beta' \leq \beta$. By the argument in part (iii) every edge of $F(G)$ lies on a cycle. The length of each cycle is finite. Thus $A$ has at most $\aleph_0, \beta' = \max\{\beta', \aleph_0\} = \beta'$ edges if $\beta'$ is infinite and it has a finite number of edges if $\beta'$ is finite. Since $|E(A)| = \beta$, which is infinite, $\beta' \geq \beta$. We conclude that $\beta' = \beta$.
vii) By parts (v, vi) \( F(G) \) has \( 2^\beta \) components and each component has \( \beta \) cycles. Since every cycle is contained in a component, \( |\mathcal{C}(F(G))| = \beta.2^\beta = 2^\beta. \)

From the above proposition it follows that an infinite graph cannot be a forest graph unless every component has the same infinite order \( \beta \) and there are \( 2^\beta \) components. A consequence is that the infinite graph itself must have order \( 2^\beta \). Hence,

**Corollary 3.2.** Any infinite graph whose order is not a power of 2, including \( \aleph_0 \) and all other limit cardinals, is not a forest graph.

**Corollary 3.3.** If \( F(G) \) is connected, then \( F(G) \) is finite.

**Proof:** Suppose that \( F(G) \) is connected. If \( G \) has infinitely many cycles then by Proposition 3.1(v) \( F(G) \) is disconnected. Therefore \( G \) has finitely many cycles. Let \( A = \{ e \in E(G) : \text{edge } e \text{ lies on a cycle in } G \} \). Then \( |A| \) is finite, say \( |A| = n \), because the length of each cycle is finite. As every maximal forest of \( G \) consists of a maximal forest of \( A \) and all the edges of \( G \) which are not in \( A \), \( G \) has at most \( 2^n \) maximal forests. Hence \( F(G) \) has a finite number of vertices and consequently is finite.

4 F-Roots

In this section we establish properties of \( F \)-roots of graphs. We begin with the question of what an \( F \)-root should be.

Since any graph \( H' \) that is isomorphic to an \( F \)-root \( H \) of \( G \) is immediately also an \( F \)-root, the number of non-isomorphic \( F \)-roots is a better question than the number of labeled \( F \)-roots. We now show in some detail that the best question is the number of non-isomorphic \( F \)-roots without isthmi. Let \( t_\beta \) be the number of non-isomorphic rooted trees of order \( \beta \). (We note that \( t_{\aleph_0} \geq 2^{\aleph_0} \), by a construction of Reinhard Diestel (personal communication, July 10, 2015): let \( P \) be a one-way infinite path whose vertices are labelled by natural numbers, with root 1; choose any subset \( S \) of \( \mathbb{N} \) and attach two edges at every vertex in \( S \), forming a rooted tree \( T_S \) (rooted at 1). Then \( S \) is determined by \( T_S \) because the vertices in \( S \) are those of degree at least 3 in \( T_S \). The number of sets \( S \) is \( 2^{\aleph_0} \), hence \( t_{\aleph_0} \geq 2^{\aleph_0} \).)

**Proposition 4.1.** Let \( G \) be a graph with an \( F \)-root of order \( \alpha \). If \( \alpha \) is finite, then \( G \) has infinitely many non-isomorphic finite \( F \)-roots. If \( \alpha \) is finite or infinite, then \( G \) has at least \( t_\beta \) non-isomorphic \( F \)-roots of order \( \beta \) for every infinite \( \beta \geq \alpha \).
Proof: Let $G$ be a graph which has an $F$-root $H$, i.e., $F(H) \cong G$, and let $\alpha$ be the order of $H$. We may assume $H$ has no isthmri and no isolated vertices unless it is $K_1$.

Suppose $\alpha$ is finite; then let $T$ be a tree, disjoint from $H$, of any finite order $n$. Identify any vertex $v$ of $H$ with any vertex $w$ of $T$. The resulting graph $H_T$ also has $G$ as its forest graph since $T$ is contained in every maximal forest of $H_T$. As the order of $H_T$ is $\alpha + n - 1$ and $n$ can be any natural number, the graphs $H_T$ are an infinite number of non-isomorphic finite graphs with the same forest graph up to isomorphism.

Suppose $\alpha$ is finite or infinite and $\beta \geq \alpha$ is infinite. Let $T$ be a rooted tree of order $\beta$ with root $w$; for instance, $T$ can be a star rooted at the star center. Attach $T$ to a vertex $v$ of $H$ by identifying $v$ with the root $w$. Denote the resulting graph by $H_T$; it is a root of $G$ of order $\beta$ because it has order $\alpha + \beta$, which equals $\beta$ because $\beta$ is infinite and $\beta \geq \alpha$. As $H$ has no isthmri, $T$ and $w$ are determined by $H_T$; therefore, if we have a non-isomorphic rooted tree $T'$ with root $w'$ (that means there is no isomorphism of $T$ with $T'$ in which $w$ corresponds to $w'$), $H_{T'}$ is not isomorphic to $H_T$. (The one exception is when $H = K_1$, which is easy to treat separately.) The number of non-isomorphic roots of $G$ of order $\beta$ is therefore at least the number of non-isomorphic rooted trees of order $\beta$, i.e., $t_\beta$.

We may conclude from Proposition 4.1 that the most interesting question about the number of $F$-roots of a graph $G$ with an $F$-root is not the total number of non-isomorphic $F$-roots (which by Proposition 4.1 cannot be assigned any cardinality) and it is not the number of a given order; it is the number of $F$-roots that have no isthmri.

Proposition 4.2. No bipartite graph $G$ has an $F$-root.

Proof: Let $G$ be a bipartite graph of order $p$ ($p \geq 2$) and let $H$ be a root of $G$, i.e., $F(H) \cong G$. Suppose $H$ has no cycle; then $F(H)$ is $K_1$, which is a contradiction. Therefore $H$ has a cycle of length $\geq 3$. It follows by Lemma 2.8 that $F(H)$ contains $K_3$, a contradiction. Hence no bipartite graph $G$ has a root.

Theorem 4.3. No infinite connected graph has an $F$-root.

Proof: This follows by Corollary 3.3.
5 F-Convergence and F-Divergence

In this section we establish a necessary and sufficient condition for F-convergence of a graph.

Lemma 5.1. Let G be a finite graph that contains a C_n (for n ≥ 4) or at least two edge disjoint triangles, then G is F-divergent.

Proof: Let G be a finite graph. By Lemma 2.11, F_m(G) contains K_m^2 as a subgraph. Therefore, as m increases the clique size of F_m(G) increases. Hence G is F-divergent.

Lemma 5.2. If |C(G)| = β where β is infinite, then G is F-divergent.

Proof: Assume |C(G)| = β (β infinite). By Proposition 3.1(vii), as 2^β < 2^{2^β} < 2^{2^{2^β}} < ⋯, it follows that |C(F(G))| < |C(F^2(G))| < |C(F^3(G))| < ⋯. Therefore, as n increases |C(F^n(G))| increases. Hence G is F-divergent.

Theorem 5.3. Let G be a graph. Then,

i) G is F-convergent if and only if either G is acyclic or G has only one cycle, which is of length 3.

ii) If G is F-convergent, then it converges in at most two steps.

Proof: i) If G has no cycle, then it is a forest and F(G) is K_1. If G has only one cycle and that cycle has length 3, then F(G) is K_3. Therefore in each case G is F-convergent.

Conversely, suppose that G has a cycle of length greater than 3 or has at least two triangles. If G has infinitely many cycles, then it follows by Lemma 5.2 that G is F-divergent. Therefore we may assume that G has a finite number of cycles. If G has a finite number of vertices, then it is finite and by Lemma 5.1 it is F-divergent. Therefore G has an infinite number of vertices. However, it can have only a finite number of edges that are not isthmi, because each cycle is finite. Thus G consists of a finite graph G_0 and any number of isthmi and isolated vertices. Since F(G) depends only on the edges that are not isthmi and the vertices that are not isolated, F(G) = F(G_0) (under the natural identification of maximal forests in G_0 with their extensions in G by adding all isthmi of G). Therefore, G is F-divergent.

ii) If G has no cycle, then G is a forest and F(G) ∼= F^2(G) ∼= K_1. If G has only one cycle, which is of length 3, then F(G) ∼= F^2(G) ∼= K_3. Therefore G converges in at most 2 steps.
Corollary 5.4. A graph $G$ is $F$-stable if and only if $G = K_1$ or $K_3$.

6 F-Depth

In this section we establish results about the $F$-depth of a graph.

Theorem 6.1. Let $G$ be a finite graph. The $F$-depth of $G$ is finite if and only if $G$ is different from $K_1$ and $K_3$.

Proof: Let $G$ be a finite graph. Suppose that $G$ is $K_1$ or $K_3$. Then by Corollary 5.4, it follows that $G$ is $F$-stable. Therefore, the $F$-depth of $G$ is infinite.

Conversely, suppose that $G$ is different from $K_1$ and $K_3$.

Case 1: Let $|V| < 4$. Then $G$ has no $F$-root so its $F$-depth is zero.

Case 2: Let $|V| = 4$. Suppose $G$ has an $F$-root $H$ (i.e., $F(H) \cong G$). Then $H$ should have exactly 4 maximal forests. That is possible only when $H$ has only one cycle, which is of length 4. By Lemma 2.8 it follows that $F(H)$ contains $K_4$, hence it is $K_4$. Therefore $G$ has an $F$-root if and only if it is $K_4$. Hence the $F$-depth of $G$ is zero, except that the depth of $K_4$ is 1.

Case 3: Let $|V| = n$ where $n > 4$. Suppose that $G$ has infinite $F$-depth. Then for every $n$ there is a graph $H_n$ such that $F^n(H_n) = G$. If $H_n$ does not have two triangles or a cycle of length greater than 3, then $H_n$ has only one cycle which is of length 3, or no cycle and $H_n$ converges to $K_1$ or $K_3$ in at most two steps, a contradiction. Therefore $H_n$ has two triangles or a cycle of length greater than 3. By Lemma 2.11 it follows that $F^i(H_n)$ contains $K^2_i$ for each $i > 2$, so that $F^n(H_n)$ contains $K^n_2$. That is, $G$ contains $K^n_2$. This is impossible as $G$ has order $n$. Hence the $F$-depth of $G$ is finite.

Theorem 6.2. The $F$-depth of any infinite graph is finite.

Proof: Let $G$ be a graph of infinite order $\alpha$. If $G$ has a root, then $G$ is without isthmi or isolated vertices.

If $G$ is connected, Theorem 4.3 implies that $G$ has no root. Therefore its $F$-depth is zero.

If $G$ is disconnected, assume it has infinite depth. Then for each natural number $n$ there exists a graph $H_n$ such that $G \cong F^n(H_n)$. Let $\beta_n$ denote the order of $H_n$. Since $F(H_1) \cong G$, by Proposition 3.1(ii) $\alpha = 2^{\beta_1}$, from which we infer that $\beta_1 < \alpha$. 

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This is independent of which root $H_1$ is, so in particular we can take $H_1 = F(H_2)$ and conclude that $\beta_1 = 2^{\beta_2}$, hence that $\beta_2 < \beta_1$. Continuing in like manner we get an infinite decreasing sequence of cardinal numbers starting with $\alpha$. The cardinal numbers are well ordered [8], so they cannot contain such an infinite sequence. It follows that the $F$-depth of $G$ must be finite.

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References


