2. T. Zaslavsky: Voltage-graphic geometry and the forest lattice (chaired by N. Robertson)

1. We begin with a theorem that provides a focal point for the general theory. Let $\Gamma = (N,E)$ be a graph, $n = |N|$, $f_k = \text{the number of } k\text{-tree spanning forests in } \Gamma$, and $t(\Delta) = \text{the number of tree components of the graph } \Delta$. Let $\mathcal{F}$ be the set of forests of $\Gamma$, including the null graph, ordered in the following way: $F \leq F'$ if $F'$ consists of some (or no) trees of $F$ plus (optionally) additional edges linking some of these trees.

**Forest Theorem.** $\mathcal{F}$ is a geometric lattice of rank $n$. Its rank function is $\text{rk } F = n - t(F)$. Its characteristic polynomial (when $\Gamma$ is finite) is

$$p_{\mathcal{F}}(\lambda) = (-1)^n \sum_{k=0}^{n} (1 - \lambda)^k f_k.$$

Some other facts about $\mathcal{F}$: its 0 element is $(N,\emptyset)$, its 1 is $(\emptyset,\emptyset)$, its atoms are $(N,e)$ for each link $e$ and $(N\setminus\{v\}, \emptyset)$ for each vertex $v$.

The Forest Theorem can be proved directly, e.g. by deletion-contraction, but it is more interesting to derive it from the theory of voltage-graphic matroids.
2. A **voltage graph** is a pair \((\Gamma, \varphi)\) consisting of a graph \(\Gamma = (N, E)\) and a **voltage**, a mapping \(\varphi: E \to G\) where \(G\) is a group called the **voltage group**. The voltage on an edge depends on the sense in which the edge is traversed: if for \(e\) in one direction the voltage is \(\varphi(e)\), then in the opposite direction it is \(\varphi(e)^{-1}\). The voltage on a circle is the product of the edge voltages taken in order with consistent direction; if the product equals 1 the circle is called **balanced**. (While in general the starting point and orientation of \(C\) influence its voltage, they have no effect on whether it is balanced.)

A subgraph is balanced if every circle in it is balanced. For \(S \subseteq E\), let \(b(S) = \text{the number of balanced components of } (N, S)\).

**Matroid Theorem.** The function \(rk S = n - b(S)\) is the rank function of a matroid \(G(\Gamma, \varphi)\) on the set \(E\). A set \(A \subseteq E\) is closed iff every edge \(e \notin A\) has an endpoint in a balanced component of \((N, A)\) but does not combine with edges in \(A\) to form a balanced circle. A set is a circuit iff it is a balanced circle or a bicircular graph containing no balanced circle.

**Bicircular graphs**

**Theta graphs**

**Handcuffs**
We call \( G(\Gamma, \varphi) \) a **voltage-graphic matroid**. In case it is a simple matroid it is a subgeometry of the Dowling lattice \( Q_n(\Theta) \) (see [1]).

**Example 1.** \( \varphi = 1 \). Then \( G(\Gamma, \varphi) = G(\Gamma) \), the usual graphic matroid.

**Example 2.** \( \Theta = \{+1\} \). Then \((\Gamma, \varphi)\) is a signed graph.

**Example 2a.** Same, with \( \varphi = -1 \). Then \( G(\Gamma, \varphi) \) is the even-circle matroid of \( \Gamma \) (see [2] for references).

**Example 3.** No balanced circles. Then \( G(\Gamma, \varphi) = B(\Gamma) \), the bicircular matroid of \( \Gamma \) (see [4] for references). The balanced sets are the spanning forests. The closed sets correspond to the forests \( F = (X, E(F)) \) such that the subgraph of \( \Gamma \) induced on \( X^c \) has no trees. The circuits are the bicircular graphs (whence the name). The rank function is \( \text{rk } S = n - t(S) \).

The first parts of the Forest Theorem follow from these observations, the Matroid Theorem, and:

**Lemma.** \( \mathcal{F} \) = the lattice of flats of \( B(\Gamma^c) \), where \( \Gamma^c \) denotes \( \Gamma \) with a loop at every node.

3. Now let \( \Gamma \) be finite and let \( \Theta \) have finite order \( g \). A proper \( \mu \)-coloring of \((\Gamma, \varphi)\) is a mapping

\[ \kappa : N \rightarrow \{0\} \cup \{1, \ldots, \mu\} \times \Theta \]
such that, for any edge $e$ from $v$ to $w$ (including loops), we have $\kappa(v) \neq 0$ or $\kappa(w) \neq 0$ and also

$$\kappa_1(v) \neq \kappa_1(w) \text{ or } \kappa_2(w) \neq \kappa_2(v) \varphi(e) \quad \text{if } \kappa(v), \kappa(w) \neq 0,$$

where $\kappa_1$ and $\kappa_2$ are the numerical and group parts of $\kappa$. Let $\chi(\mu g + 1)$ be the number of proper $\mu$-colorings of $(\Gamma, \varphi)$ and let $\chi^b(\mu g)$ be the number which do not take the value 0.

**Chromatic Polynomial Theorem.** $\chi(\mu g + 1)$ is a polynomial in $\mu$.
Indeed $\chi(\lambda) = \lambda^b(\mu g) p(\lambda)$, where $p(\lambda)$ is the characteristic polynomial of $G(\Gamma, \varphi)$.

**Balanced Chromatic Polynomial Theorem.** $\chi^b(\mu g)$ is a polynomial in $\mu$. Indeed $\chi^b(\lambda) = \sum S \lambda^b(S)(-1)^{|S|}$, summed over balanced $S \subseteq E$.

**Fundamental Theorem.** Let $\chi^b_X(\lambda)$ denote the balanced chromatic polynomial of the induced voltage graph on $X \subseteq N$. Then

$$\chi(\lambda) = \sum_{X \text{ stable}} \chi^b_X(\lambda - 1).$$

In particular for the forest lattice we look at $B(\Gamma^C)$. The necessary finite voltage group may be, for instance, the power set $\mathcal{P}(E)$ with symmetric difference, with voltage $\varphi(e) = \{e\}$. Then the latter two theorems quickly yield the characteristic polynomial of $\mathfrak{F}$.

